Semigroups of inverse quotients

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Groups of Left Quotients

Definition

Let S be a subsemigroup of a group G. Then G is a **group of left quotients** of S if every $g \in G$ can be written as $g = a^{-1}b$ for some $a, b \in S$.

Example (Constructing the Integers from the Natural Numbers)

Let $G = (\mathbb{Z}, +)$ be the group of integers under addition, and let $S = (\mathbb{N}, +)$ be the subsemigroup of natural numbers.

We see that every $g \in G$ can be written as $g = n - m = m^{-1}n$ for some $m, n \in S$, so G is a group of left quotients of S.

Monoids of Classical Left Quotients

Definition

Let S be a subsemigroup of a monoid M. Then M is a **monoid of classical left quotients** of S if

(i) for every cancellative $a \in S$, there exists an $a^{-1} \in M$

(ii) every $m \in M$ can be written as $m = a^{-1}b$ for some $a, b \in S$.

Example (Constructing the Rationals from the Integers)

Let $Q = (\mathbb{Q}, \times)$ be the monoid of rationals under multiplication, and let $Z = (\mathbb{Z}, \times)$ be the subsemigroup of integers.

We see that every $q \in Q$ can be written as $q = \frac{n}{m} = m^{-1}n$ for some $m, n \in Z$, so Q is a monoid of classical left quotients of Z.

Semigroups of Left Fountain-Gould Quotients

Definition

Let S be a subsemigroup of a semigroup Q. Then Q is a **semigroup of left Fountain-Gould quotients** of S, and S is a **left Fountain-Gould order** in Q if

(i) every square-cancellable element lies in a subgroup of Q

(ii) every $q \in Q$ can be written as $q = a^{\#}b$ for some $a, b \in S$, where $a^{\#}$ denotes the inverse of a in some subgroup of Q.

Moreover if every every $q \in Q$ can be written as $q = a^{\#}b$, where $a \mathcal{R}^{Q} b$, we say that Q is **straight** over S, or S is **straight** in Q.

Semigroups of left I-quotients

Definition

Let S be a subsemigroup of an inverse semigroup Q.

Then Q is a **semigroup of left I-quotients** of S, and S is a **left I-order** in Q if every $q \in Q$ can be written as $q = a^{-1}b$ for some $a, b \in S$,
where a^{-1} denotes the inverse in the sense of inverse semigroup theory.

Moreover if every every $q \in Q$ can be written as $q = a^{-1}b$, where $a \mathcal{R}^Q b$, we say that Q is **semigroup of straight left I-quotients** of S, or S is **straight left I-order** in Q.

Example (Bicyclic Monoid)

Let $B = \langle a, b \mid ab = 1 \rangle$ be the Bicyclic Monoid and $S = \{a^n \mid n \in \mathbb{N}^0\}$.

B is an inverse semigroup with $(b^m a^n)^{-1} = (b^n a^m)$, and every element in B can be expressed as $b^m a^n = (a^m)^{-1} a^n$, so B is a semigroup of left I-quotients of S.

Additionally, since S is contained within a single $\mathcal{R}^B$-class, we see that S is straight in B.

Left I-orders intersecting every $\mathcal{L}$-class

Lemma (Gould)

Let S be a left I-order in Q.
Then S is straight in Q if and only if S intersects every $\mathcal{L}$-class of Q.

Proof.

Let S be straight in Q, and let $q = a^{-1}b \in Q$ such that $a, b \in S$ and $a \, \mathcal{R}^Q \, b$. Then

$$q^{-1}q = b^{-1}aa^{-1}b = b^{-1}bb^{-1}b = b^{-1}b,$$

and so $b \in S \cap L_q$.

Proof (cont.)

Conversely, suppose S intersects every $\mathcal{L}$-class of Q.

Let $q \in Q$; we know that $q = a^{-1}b$, where $a, b \in S$. Then

$$q = a^{-1}aa^{-1}bb^{-1}b = a^{-1}fb,$$

where $f = aa^{-1}bb^{-1} \in E(Q)$.

Since S intersects every $\mathcal{L}$-class, there exists $u \in S \cap L_f$, and so $f = u^{-1}u$. Hence

$$(ua)(ua)^{-1} = uaa^{-1}u^{-1} = ufaa^{-1}u^{-1} = ufu^{-1} = uu^{-1}.$$

Similarly $(ub)(ub)^{-1} = uu^{-1}$.
We can therefore write

$$q = a^{-1}fb = a^{-1}u^{-1}ub = (ua)^{-1}(ub),$$

where $ua \, \mathcal{R}^Q \, ub$. It follows that Q is straight over S. □

Finite left I-orders are straight

Theorem (S.)

Let S be a left I-order in a finite inverse semigroup, Q. Then S is straight in Q.

Proof.

By the Vagner-Preston Theorem, we know that $Q \leq \mathcal{I}_n$ for some n. Therefore

$$\begin{aligned} a\,\mathcal{R}^Q\,b &\iff aa^{-1} = bb^{-1} \iff \text{dom}(a) = \text{dom}(b), \\ a\,\mathcal{L}^Q\,b &\iff a^{-1}a = b^{-1}b \iff \text{im}(a) = \text{im}(b), \\ a \leq_{\mathcal{R}^Q} b &\iff aa^{-1}bb^{-1} = aa^{-1} \iff \text{dom}(a) \subseteq \text{dom}(b), \\ a \leq_{\mathcal{L}^Q} b &\iff a^{-1}ab^{-1}b = a^{-1}a \iff \text{im}(a) \subseteq \text{im}(b). \end{aligned}$$

Proof (cont.)

We define the rank of an element $a \in Q$ as

$$\mathrm{rank}(a) = |\mathrm{dom}(a)| = |\mathrm{im}(a)|.$$

We can extend the definition of rank to $\mathcal{R}$-classes and $\mathcal{L}$-classes as

$$\mathrm{rank}(R_a) = \mathrm{rank}(L_a) = \mathrm{rank}(a).$$

This is obviously well-defined.

Proof (cont.)

Assume that S is not straight in Q.

Then S does not intersect every $\mathcal{L}$-class of Q.

Let L be an $\mathcal{L}$-class of Q that does not intersect S of largest possible rank.

Let the rank of L be k, and let e be the idempotent in L.

Since S is an I-order in Q, we know that there exists $a, b, \in S$ such that

$$e = a^{-1}b.$$

Since e is an idempotent, we have that

$$e = e^{-1} = b^{-1}a \leq_{\mathcal{L}^Q} a.$$

Therefore $\mathrm{im}(e) \subseteq \mathrm{im}(a)$.

Since L doesn't intersect S, we know that $e \not\mathcal{L} a$.

Therefore $\mathrm{im}(a) \neq \mathrm{im}(b)$, and so $\mathrm{im}(e) \subset \mathrm{im}(a)$ strictly.

Hence $\mathrm{rank}(a) > \mathrm{rank}(e) = k$.

Proof (cont.)

We know that $\mathrm{dom}(aa^{-1}) = \mathrm{dom}(a)$. Therefore

$$\mathrm{rank}(aa^{-1}) = \mathrm{rank}(a) > k.$$

Since aa^{-1} is of larger rank than k, we know that $L_{aa^{-1}} \cap S \neq \emptyset$.
Let $s \in L_{aa^{-1}} \cap S$. Then

$$e = a^{-1}b \,\mathcal{L}\, aa^{-1}b = s^{-1}sb \,\mathcal{L}\, sb.$$

Therefore $sb \in L \cap S$. Contradiction. □

Fin