Direct and subdirect products in combinatorial semigroup theory

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Once again I fall into...

(Ásdís Sif Gunnarsdóttir; Ragnar Kjartansson)

DIRECT PRODUCTS

...the fundamental operation of this loosening method is, if one writes, fragmentation, and, if one teaches, digression, or [...] excursion. [...] comings and goings of a child playing [...] within which the pebble, the string come to matter less than the enthusiastic giving of them. (R. Barthes)

Very little to define...

$$S \times T = \{(s,t) \,:\, s \in S,\ t \in T\},\ (s,t)(u,v) = (su, tv)$$

$$S_1 \times \cdots \times S_n = \{(s_1, \dots, s_n) \,:\, s_i \in S_i\}$$

$$\prod_{i \in I} S_i = \{f : I \to \bigcup_{i \in I} S_i \,:\, f(i) \in S_i\},\ (fg)(i) = (f(i))(g(i)).$$

At times I will digress about other, or indeed general, algebraic structures: component-wise operations

Two basic facts

1. Always: S, T are homomorphic images of $S \times T$, via natural projections:

$$\pi_S : S \times T \twoheadrightarrow S,\ (s,t) \mapsto s$$
$$\pi_T : S \times T \twoheadrightarrow T,\ (s,t) \mapsto t$$

2. Sometimes: S, T naturally embed into $S \times T$, e.g. when S, T are monoids:

$$\iota_S : S \hookrightarrow S \times T,\ s \mapsto (s, 1_T)$$
$$\iota_T : T \hookrightarrow S \times T,\ t \mapsto (1_S, t)$$

In many situations this gives results of the form:
$S \times T$ satisfies property $\mathcal{P}$ if and only if both S and T satisfy $\mathcal{P}$.

Finite generation (f.g.)

Theorem (folklore)

For S, T – monoids: $\quad S \times T$ *f.g.* $\Leftrightarrow S, T$ *f.g.*

($\Rightarrow$) F.g. preserved by homomoprhic images.

($\Leftarrow$) Copies of S, T in $S \times T$ generate $S \times T$.

But: $\mathbb{N} \times \mathbb{N}$ not finitely generated: $(1, n)$, $n \in \mathbb{N}$ indecomposable.

Theorem (Robertson, NR, Wiegold 1998)

For S, T infinite semigroups: $S \times T$ is finitely generated if and only if both S, T are finitely generated and they don't have any indecomposable elements ($SS = S$, $TT = T$).

Excursion: general (=universal) algebra

With Peter Mayr, Colorado, Boulder: combinatorial theory of direct products in general algebra

Variety: class of algebras defined by identities

Congruence permutable (CP): $\rho \circ \sigma = \sigma \circ \rho$ for all congruences σ, ρ

CP varieties include: groups, rings, associative and Lie algebras, modules, loops,. . . ; but exclude semigroups, lattices.

Theorem (P. Mayr, NR 2018)

For $\mathcal{V}$ – CP variety; $A, B \in \mathcal{V}$: $A \times B$ f.g. $\Leftrightarrow$ A, B f.g.

Remark

$\mathcal{V}$ is CP iff it has a Malcev term: $m(x, y, y) = m(y, y, x) = x$.

Residual finiteness

Definition

S is residually finite (RF) if for any distinct $s, t \in S$ there exists a homomorphism $f : S \to T$ into a finite semigroup T such that $f(s) \neq f(t)$. [Elements can be separated in finite quotients.]

Theorem (folklore)

For S, T – monoids: $\quad S \times T$ *is RF* $\Leftrightarrow$ S, T *are RF.*

($\Leftarrow$) true in any algebraic structures.
($\Rightarrow$) RF preserved by substructures.

Theorem (Gray, NR 2009)

For S, T – semigroups: $\quad S \times T$ *is RF* $\Leftrightarrow$ S, T *are RF.*

Residual finiteness elsewhere in algebra

Theorem (Mayr, NR 2018)

For $\mathcal{V}$ – a congruence modular variety, $A, B \in \mathcal{V}$:

$$A \times B \text{ is RF} \Leftrightarrow A, B \text{ are RF.}$$

Not true for general algebras: Gray, NR (2009) had examples of unary algebras. Bill de Witt characterised RF of direct products of monounary algebras.

B. de Witt, Residual Finiteness and Related Properties in Monounary Algebras and their Direct Products, Algebra Universalis 82 (2021)

Separability properties

G. O'Reilly, M. Quick, NR, On separability properties in direct products of semigroups, Monatsh. Math. 197 (2022).

Four separation properties: 1. complete separability, 2. strong subsemigroup separability, 3. weak subsemigroup separability, 4. monogenic subsemigroup separability.

Definition

S is strongly subsemigroup separable (SSS) if for every $s \in S$, $U \leq S$ with $s \notin U$ there is a homomorphism $f : S \to T$, T finite, such that $f(s) \notin f(T)$. [Elements can be separated from subsemigroups in finite quotient.]

Theorem

If $S \times T$ is SSS then both S, T are SSS. The converse is not true.

SSS-preserving

G. O'Reilly, M. Quick, NR, On separability properties in direct products of semigroups, Monatsh. Math. 197 (2022).

Definition

S is SSS-preserving if for every SSS semigroup T we have that $S \times T$ is also SSS.

Theorem

A finite semigroup S is SSS-preserving if and only if every element of S is either indecomposable or belongs to a subgoup (i.e. S is an ideal extension of a completely regular semigroup by a null semigroup).

Noetherian semigroups

Definition

S is (right) noetherian if its right congruences satisfy the ascending chain condition (equivalently: all right congruences f.g.).

Theorem (folklore)

A group G is noetherian iff all its subgroups are finitely generated.

Question

Is the following true for all monoids S, T:

$$S \times T \text{ noetherian } \Leftrightarrow S, T \text{ noetherian?}$$

($\Rightarrow$) always true. The converse not true for semigroups.
True when S is: finite, inverse, commutative.

C. Miller, NR, Right noetherian semigroups, IJAC 30 (2020).

Structure?

To address questions such as these, need to have a better handle on fundamental structure of direct products:

- congruences (one-sided, two-sided)
- subsemigroups

Subdirect products

Are subsemigroups of direct products direct products of subsemigroups? No!
E.g. $\Delta_S = \{(s, s) : s \in S\} \leq S \times S$.

Definition

A subdirect product of S and T is a subsemigroup $P \leq S \times T$ such that $\pi_S(P) = S$, $\pi_T(P) = T$.

A subsemigroup of $S \times T$ is a subdirect product of subsemigroups.

Issue: how to construct them? (property vs. construction)

Fiber products

S_1, S_2 – semigroups

$\phi_i : S_i \twoheadrightarrow U$ – epimorphisms onto a common quotient

The set

$$\{(s_1, s_2) \in S_1 \times S_2 \,:\, \phi_1(s_1) = \phi_2(s_2)\}$$

is a subdirect product (called fiber product w.r.t. ϕ_1, ϕ_2).

Theorem (Goursat's Lemma)

Every subdirect product of two groups is fiber.

Generalises to: subdirect products in congruence permutable varieties (Fleischer's Lemma).

But not true for semigroups.

Tricky subdirect products of non-tricky groups

There exist subdirect products of the free product of two free groups of rank 2 that are:

- not finitely generated (Bridson, Miller III, 2009);
- finitely generated but not finitely presented (Grunewald 1978);
- finitely generated but with an undecidable membership problem (Mikhaĭlova 1966).

Commutative semigroups

Fact. There are only countably many (group) subdirect products in $\mathbb{Z} \times \mathbb{Z}$. They are all finitely generated and presented.

A. Clayton, N. Ruškuc, On the number of subsemigroups of direct products involving the free monogenic semigroup, JAuMS 109 (2020).

Theorem

There are uncountably many non-isomorphic subdirect products in $\mathbb{N} \times \mathbb{N}$.

A. Clayton, K. Reilly, NR: subdirect products in $S \times T$, where $S, T \in \{\mathbb{N}, \mathbb{N}_0, \mathbb{Z}\}$. There are uncountably many (semigroup) subdirect products in $\mathbb{Z} \times \mathbb{Z}$.

Of course, most are not fiber.

Some future directions

Project

Develop new semigroup constructions, which give subdirect products which are not necessarily fiber products.

Problem

Is it possible to classify all (finite?) semigroups S, T such that all subdirect products of S and T are fiber?

Congruences

Groups: congruences $\leftrightarrow$ normal subgroups

Theorem (folklore)

G_1, G_2 – groups. Normal subgroups of $G_1 \times G_2$ are precisely the subdirect product of $N_1 \times N_2$, $N_i \trianglelefteq G_i$, with $\phi_i : N_i \to Q$, $[G_i, N_i] \leq \ker(\phi_i)$.

J. Araújo, W. Bentz, G.M.S. Gomes (2019): congruences on direct products involving $\mathcal{T}_n$, $\mathcal{PT}_n$, $\mathcal{I}_n$.

Fig. 8 $F, \mathcal{H}, \mathcal{H}$

Fig. 9 F, F, ε

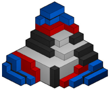


Fig. 10 $F, \mathcal{H}, \varepsilon$

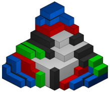


Fig. 11 $F, \varepsilon, \varepsilon$

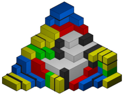


Fig. 12 $\mathcal{H}, \mathcal{H}, \mathcal{H}$

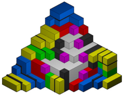
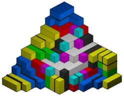


Fig. 13 $\mathcal{H}, \mathcal{H}, \varepsilon$



Congruences

Matthew Brookes, NR: the general problem of determining congruences on $S \times T$

Some success where the factors are simple – work in progress.

Christy Chao, NR: congruences of $S \times Z$, where Z is a null-semigroup.

Some observations/questions:

- Congruences on a group G are subgroups of $G \times G$ containing the diagonal $\Delta_G = \{(g,g) : g \in G\}$
- Not the case for all semigroups. Characterise when it is?
- Can one develop new conditions, viewing congruences as subdirect products?
- How does this apply to congruences of $S \times T$?
- Are one-sided congruences of $S \times T$ harder or easier to understand?

Subdirect products: several factors

Subdirect products in $S_1 \times \cdots \times S_n$.

Issue: Fiber product does not iterate well.

Theorem (Bridson, Miller III, Howie, Short 2013)

If $G_1, \ldots, G_n$ are finitely generated (resp. finitely presented) groups and P is a subdirect product in $G_1 \times \cdots \times G_n$ which is virtually surjective on pairs (VSP) then P is finitely generated (resp. finitely presented).

VSP: $[G_i \times G_j : \pi_{ij}(P)] < \infty$.

Mayr, NR (2019): generalises to rings, and K-algebras over noetherian rings, but not to general CP varieties.

Problem

Can we think of a 'surjectivity-like' condition for semigroups for which the above theorem would work?

Infinitely many factors

Less and and less coherent...

Theorem (after Hickin, Plotkin 1981; McKenzie 1982)

G – finite group. G has only countably many countable subdirect powers up to isomorphism if and only if G is abelian. Otherwise G has $\mathfrak{c} = 2^{\aleph_0}$ countable non-isomorphic subdirect powers.

Question

Classify finite semigroups with only $\aleph_0$ non-isomorphic countable subdirect powers.

Infinitely many factors

Theorem (Clayton, NR; in prep)

S – commutative semigroup. S has only $\aleph_0$ non-isomorphic countable subdirect powers if and only if S is an abelian group or a null semigroup; otherwise it has $\mathfrak{c}$.

Observation: There are non-commutative semigroups with countably many countable subdirect powers: left/right zero semigroups; rectangular bands.

Boolean powers

Boolean power:

$$\begin{array}{c} \text{a semigroup } S \\ + \\ \text{a boolean algebra } B \end{array} \rightsquigarrow S^B, \text{ a subdirect power of } S$$

Boolean separating: $B_1 \not\cong B_2 \Rightarrow S^{B_1} \not\cong S^{B_2}$

Fact. Boolean separating $\Rightarrow \mathfrak{c}$ non-isomorphic subdirect powers

Question

Classify finite boolean separating semigroups.

Classification for groups: A.B. Apps (1982), Lawrence (1981)

Bergman property

Definition
S has Bergman property if $\forall$ generating set X of S, $\exists n \in \mathbb{N}$ s.t.

$$S = \{x_1 \dots x_m \,:\, m \leq n,\ x_i \in X\}.$$

Theorem (Y. de Cornulier 2006)
If G is a finite perfect group and X an infinite set then G^X has Bergman property.

Question
For which finite semigroups S does S^X have Bergman property?
Any interesting infinite examples?

Instead of conclusion...

There is an age at which we teach what we know. Then comes another age at which we teach what we do not know; this is called research. Now perhaps comes the age of another experience: that of unlearning, of yielding to the unforeseeable change which forgetting imposes [...]

[...] no power, little knowledge, a little wisdom, and as much flavour as possible. (R. Barthes)

Thank you!