Countable subdirect powers of finite commutative semigroups

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Introduction to subdirect products

Definition (subdirect product)

A *subdirect product* of two semigroups S and T is a subsemigroup U of the direct product $S \times T$ for which the projection maps

$$\pi_S : U \to S,\ (s,t) \mapsto s,$$
$$\pi_T : U \to T,\ (s,t) \mapsto t,$$

onto S and T are surjections.

Examples of subdirect products

- The direct product $S \times T$ is a subdirect product of semigroups S and T.
- $\Delta_S := \{(s,s) : s \in S\}$ is the diagonal subdirect product of a semigroup S with itself.
- Let F be the group with presentation

$$\left\langle x, y \mid [xy^{-1}, x^{-1}yx] = [xy^{-1}, x^{-2}yx^2] = 1 \right\rangle .$$

Then $\left\langle (x, y^{-1}), (y, x), (x^{-1}, x^{-1}), (y^{-1}, y) \right\rangle$ is a subdirect product of F with itself, which is not equal to $F \times F$ or Δ_F.

Subdirect powers

We can equally define subdirect products of more than two semigroups, and indeed on a countably infinite number of semigroups by viewing the Cartesian product as a set of countably infinite tuples in the following way;

$$\prod_{i \in \mathbb{N}} S_i = \{(s_1, s_2, s_3 \ldots) : s_i \in S_i \text{ for } i \in \mathbb{N}\}.$$

If the sets S_i are all equal to the same set S, we will instead refer to the above as the *Cartesian power*, denoted

$$S^{\mathbb{N}} = \{(s_1, s_2, s_3, \ldots) : (\forall i \in \mathbb{N})(s_i \in S)\}$$

Subdirect powers

A *direct power of a semigroup* S is a semigroup $S^{\mathbb{N}}$, with componentwise multiplication

$$(s_1, s_2, s_3 \ldots)(t_1, t_2, t_3 \ldots) = (s_1t_1, s_2t_2, s_3t_3, \ldots)$$

A *subdirect power* of a semigroup S is a subsemigroup U of $S^{\mathbb{N}}$ for which the projection maps onto each component are surjections.

Note: As $S^{\mathbb{N}}$ might have uncountably many elements in it, I will only be focusing on those U that have countably many elements.

Subdirect powers of finite groups

Theorem - Hickin, Plotkin (1981)

A finitely generated non-abelian group G has uncountably many subdirect powers (which are groups) up to isomorphism.

Theorem - McKenzie (1982)

A non-abelian group G has 2^{κ} non-isomorphic subdirect powers of cardinality κ, for every infinite cardinal $\kappa \geq |G|$.

If G is finite abelian, it will have countably many subdirect powers up to isomorphism.

Subdirect powers of finite groups

Theorem

A finite group G has countably many subdirect powers up to isomorphism if and only if G is abelian.

We'd like to work towards analogous results for subdirect powers of finite semigroups, that look like

Theorem(s)

A finite semigroup S has countably many non-isomorphic subdirect powers if and only if S satisfies

$\star$ *fascinating semigroup properties* $\star$

Subdirect powers of finite semigroups

For this talk, we will concentrate on finite **commutative** semigroups.

Definition

A finite commutative semigroup S will be called *countable type* if it has only countably many subdirect powers up to isomorphism.

Otherwise, it will be called *uncountable type* if it has uncountably many such.

Some small examples

Firstly, the trivial semigroup of course is countable type, because $\{1\}^{\mathbb{N}} = \{(1, 1, \dots)\} \cong \{1\}$.

The commutative semigroups of order 2 up to isomorphism are

- ⋆ $\mathbb{Z}_2$ - countable type, abelian group;
- ⋆ O_2 - countable type, as any subdirect power of O_2 is a zero semigroup, and any bijection between two is an isomorphism.
- ⋆ $U_1 = \{0, 1\}$, the two element semilattice..we will see is uncountable type.

Semilattices and orderings

⋆ Any semilattice can be viewed as an ordered set with the ordering

$$s \leq t \Leftrightarrow st = s.$$

⋆ Moreover, any *linearly* ordered set $(L, \leq)$ can be viewed as a semilattice by defining the multiplication on L to be

$$l_1 \wedge l_2 = \min\{l_1, l_2\}.$$

⋆ Two ordered sets are order isomorphic if and only if they are isomorphic as semilattices.

The case for U_1

$U_1^{\mathbb{N}}$ is a semilattice, and can be considered as an ordered set via

$$(u_1, u_2, \dots) \leq (v_1, v_2, \dots) \Leftrightarrow (\forall i \in \mathbb{N})(u_i \leq v_i).$$

Theorem - Cantor

$\mathbb{Q}$ (as a linearly ordered set) contains uncountably many linear suborders up to order isomorphism.

Strategy to find type of U_1:

- ⋆ Find an order isomorphic copy of $\mathbb{Q}$ in $U_1^{\mathbb{N}}$;
- ⋆ This implies uncountably many subsemilattices of $U_1^{\mathbb{N}}$ (u.t.i);
- ⋆ Make each of these a subdirect power.

The case for U_1

A quick side definition:

Definition

For a finite tuple $s = (s_1, s_2, \ldots, s_n) \in S^n$, we will denote by $\overline{s}$ the countably infinite tuple

$$\overline{s} = (s_1, s_2, \ldots, s_n, s_1, s_2 \ldots, s_n, s_1, \ldots) \in S^{\mathbb{N}}.$$

An element t of $S^{\mathbb{N}}$ is said to be *recurring* if $t = \overline{s}$ for some finite tuple in S^n, for some n.

Similarly, a subset of $S^{\mathbb{N}}$ is said to be recurring if all of its elements are recurring.

The case for U_1

Lemma

For two recurring elements $s, t \in U_1^{\mathbb{N}}$ with $s \leq t$, there exists a recurring $u \in U_1^{\mathbb{N}}$ with $u \neq s$, $u \neq t$, but $s \leq u \leq t$.

Corollary

$U_1^{\mathbb{N}}$ contains an order isomorphic copy of $\mathbb{Q}$, consisting of recurring elements .

The case for U_1

Lemma

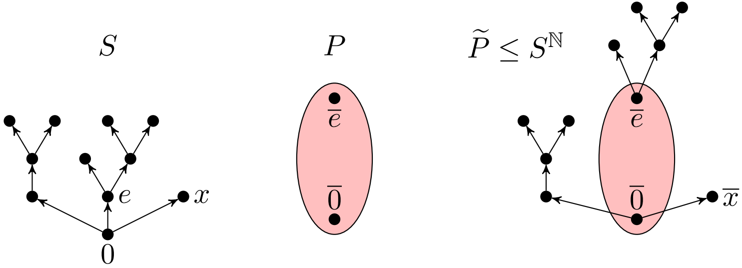
$U_1^{\mathbb{N}}$ contains uncountably many semilattices consisting of recurring elements, up to isomorphism.

A subdirect power can be constructed from each one by adding in $\overline{1}$ and $\overline{0}$, and any two non-isomorphic semilattices will give non-isomorphic subdirect powers with this construction.

This shows that U_1 is of uncountable type.

Semilattices

We can exploit U_1 to show that all (non-trivial) semilattices are uncountable type.



Theorem

Any non-trivial semilattice Y is of uncountable type.

Semilattices ⇒ Semigroups with $E(S) > 1$

For finite commutative semigroups S with $E(S) > 1$:

- ⋆ Such semigroups S are unions of "Archimedean components", which form a semilattice.
- ⋆ If $E(S) > 1$, this semilattice is non-trivial.
- ⋆ Make uncountably many subdirect powers of the semilattice, then "inflate" these to subdirect powers of S (making tuple component replacements)

Theorem

Any finite commutative semigroup S wih $E(S) > 1$ is of uncountable type.

Semigroups with a unique idempotent

That just leaves semigroups with a unique idempotent to consider.

Lemma

Let S be a finite commutative semigroup with a unique idempotent. Then S is either a group, or an ideal extension of a group by a k-nilpotent semigroup.

The case where S is a group has been dealt with. So it remains to consider ideal extensions of groups by k-nilpotent semigroups.

Semigroups with a unique idempotent

Theorem

Finite commutative k-nilpotent semigroups are of uncountable type for $k \geq 3$.

Corollary

Ideal extensions of non-trivial groups by k-nilpotent semigroups for $k \geq 2$ are of uncountable type.

Theorem (C, Ruškuc, 2021)

A finite commutative semigroup S is of countable type if and only if S is either a group, or a zero semigroup.

Further questions and results

What are the types of non-commutative completely simple semigroups? What about finite semigroups in general? Other algebras?

Theorem (Ruškuc, Witt, 2021)

Let $A = (A, \mathcal{F})$ be a finite unary algebra. The number of non-isomorphic subdirect powers of A is countable if and only if every operation in F is either a bijection or a constant mapping.

Thank you for listening!