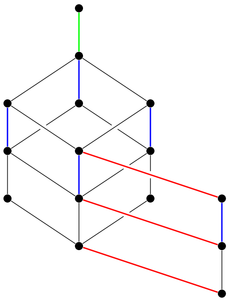
Semigroups generated by idempotents and one-sided units



James East

NBSAN 30 — Manchester — 12 July 2019

Outline

- (Partial) Brauer monoids
  - Submonoids generated by combinations of idempotents and one-/two-sided units
- Monoids
  - Lattices of submonoids
  - A semigroup of functors
    - Or: A monoid of monoidal functors on the monoidal category of monoids

Brauer monoids

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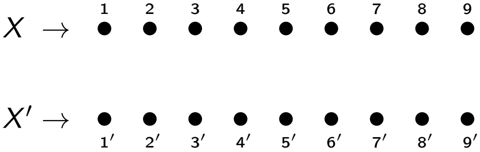

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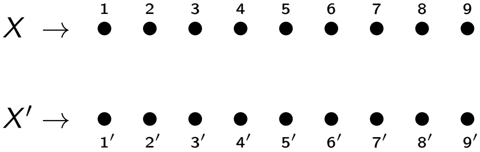
- Let n be a positive integer.
- Write $X = \{1, \dots, n\}$ and $X' = \{1', \dots, n'\}$.
- Let $\mathcal{B}_n = \{\text{matchings of } X \cup X'\}$



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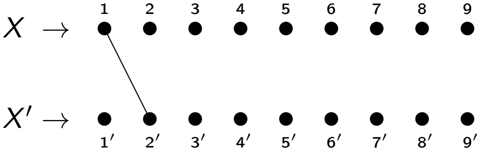

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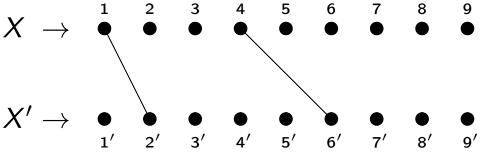

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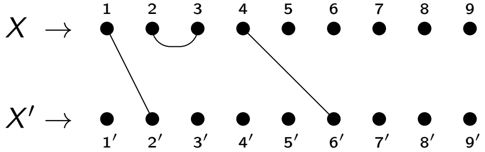

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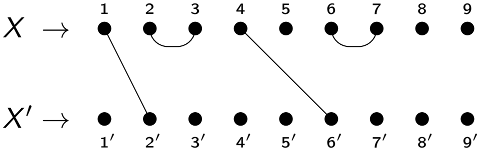

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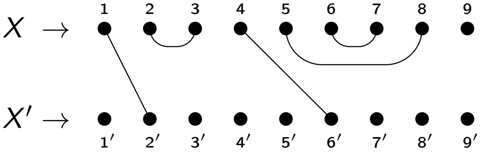

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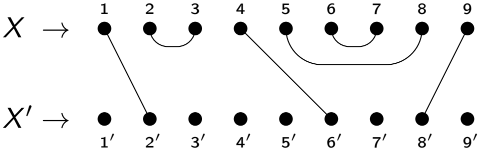

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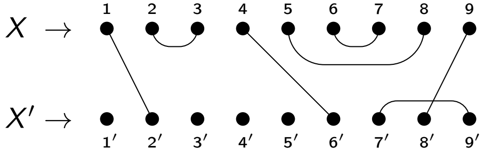

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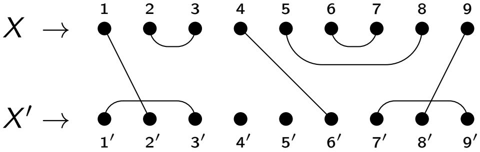

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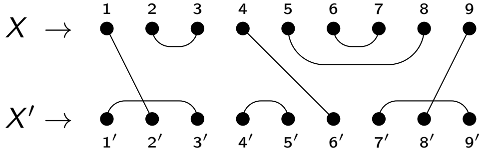

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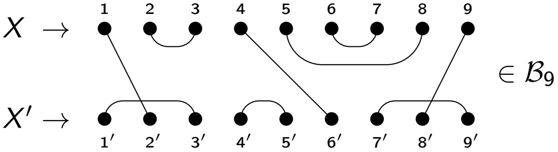

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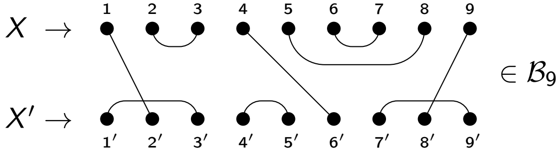
  $= \text{the Brauer monoid of degree } n.$



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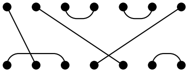
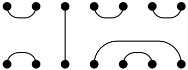
Let $\alpha, \beta \in \mathcal{B}_n$.

$\alpha =$ 

$\beta =$ 

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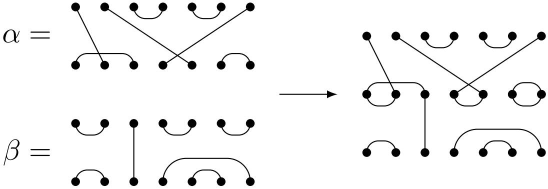
Let $\alpha, \beta \in \mathcal{B}_n$. To calculate $\alpha\beta$:

(1) connect bottom of α to top of β,



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Let $\alpha, \beta \in \mathcal{B}_n$. To calculate $\alpha\beta$:

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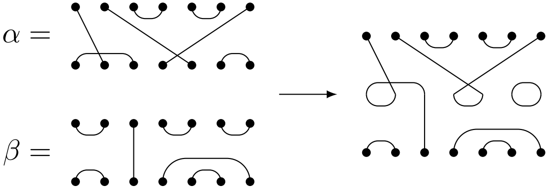

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Let $\alpha, \beta \in \mathcal{B}_n$. To calculate $\alpha\beta$:

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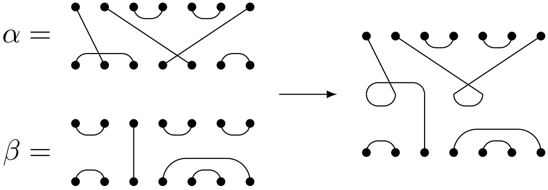

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Let $\alpha, \beta \in \mathcal{B}_n$. To calculate $\alpha\beta$:

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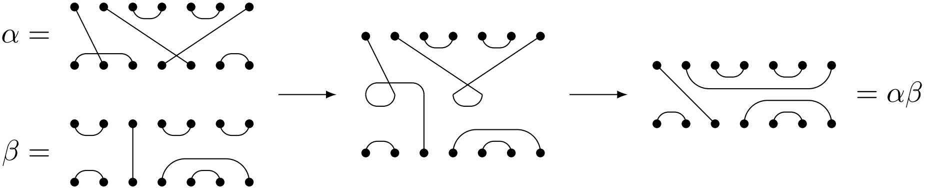
(3) smooth out resulting graph to obtain $\alpha\beta$.



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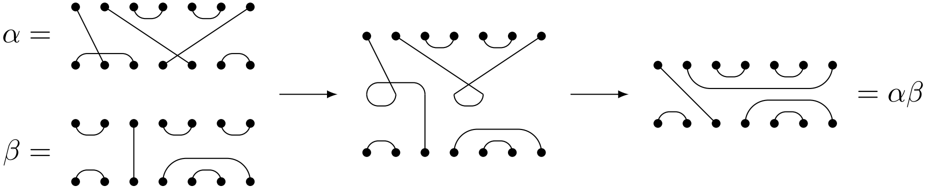

The operation is associative, so $\mathcal{B}_n$ is a semigroup (monoid, etc).

Brauer algebras — some history

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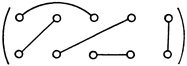
Every term of this polynomial must contain each of the vectors $\mathfrak{u}(1), \mathfrak{u}(2), \cdots, \mathfrak{u}(f), \mathfrak{x}(1), \mathfrak{x}(2), \cdots, \mathfrak{x}(f)$ exactly once. Therefore, J is a linear combination of the products of the form,

(38) $$J = (\mathfrak{v}(1), \mathfrak{v}(2))(\mathfrak{v}(3), \mathfrak{v}(4)) \cdots (\mathfrak{v}(2f-1), \mathfrak{v}(2f)),$$

where $\mathfrak{v}(1), \mathfrak{v}(2), \cdots, \mathfrak{v}(2f)$ form a permutation of $\mathfrak{u}(1), \cdots, \mathfrak{u}(f), \mathfrak{x}(1), \cdots, \mathfrak{x}(f)$. We represent $\mathfrak{u}(1), \mathfrak{u}(2), \cdots, \mathfrak{u}(f)$ by f dots in a row, and $\mathfrak{x}(1), \mathfrak{x}(2), \cdots, \mathfrak{x}(f)$ by f dots in a second row. We connect two dots by a line, if the inner product of the corresponding vectors appears in (38). We thus obtain symbols S of the following type (e.g. $f = 5$)

(39)



To every such symbol S corresponds an invariant (38) which will be denoted by J_S. For instance, the symbol (39) corresponds to

(40) $$(\mathfrak{u}(1), \mathfrak{u}(3))(\mathfrak{u}(2), \mathfrak{x}(1))(\mathfrak{u}(4), \mathfrak{x}(2))(\mathfrak{u}(5), \mathfrak{x}(5))(\mathfrak{x}(3), \mathfrak{x}(4)).$$

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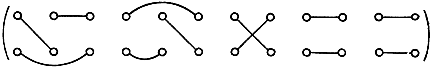
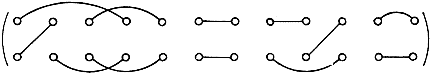
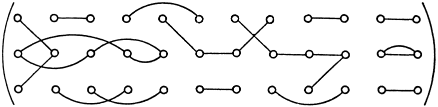
$S_1 =$ 

$S_2 =$ 

we obtain

(43) 

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It may happen that the N elements S are linearly dependent in B. We consider the N symbols S as basis elements of a new algebra Γ of order N and define multiplication by (44). Then B is a representation of Γ (but not necessarily a (1-1)-representation). It is easy to show that Γ is associative.

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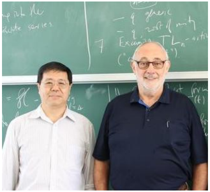

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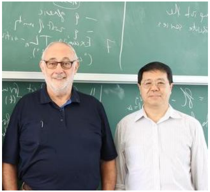

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- They've been studied intensively ever since.

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- Inspiration/techniques taken from transformation semigroups

Infinite Brauer monoids

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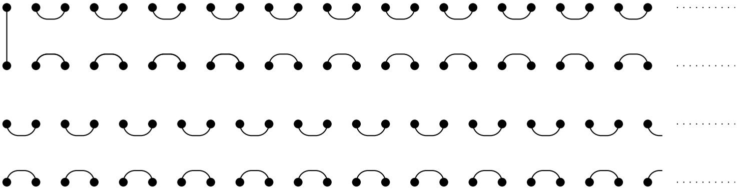
- Consider $\alpha, \beta \in \mathcal{B}_{\mathbb{N}}$ below.

$\alpha =$

$\beta =$



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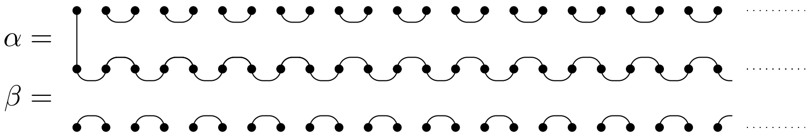

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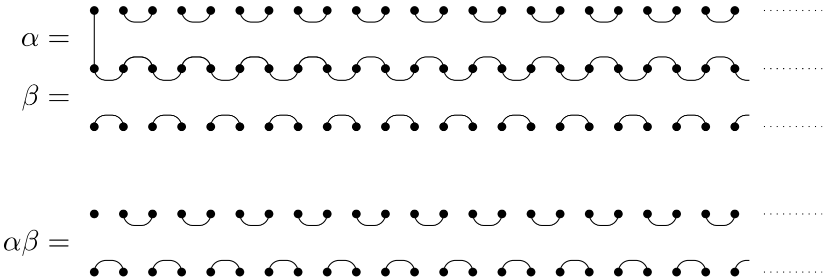

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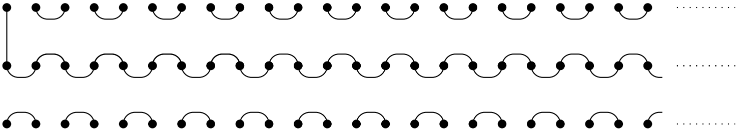

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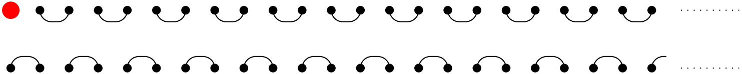

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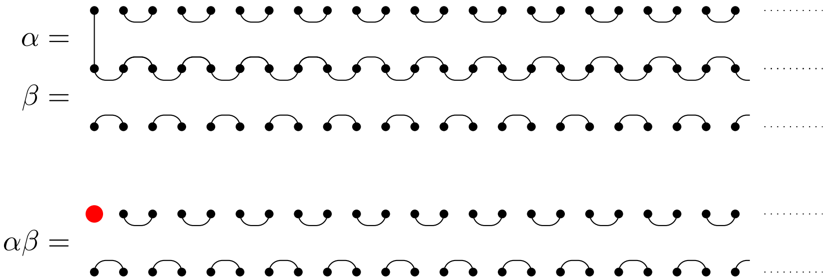
- Consider $\alpha, \beta \in \mathcal{B}_{\mathbb{N}}$ below.
- $\alpha\beta \notin \mathcal{B}_{\mathbb{N}}$!



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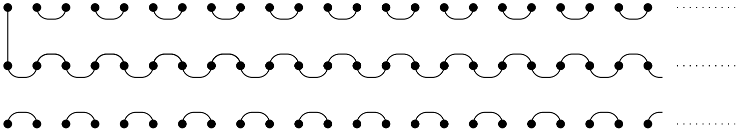
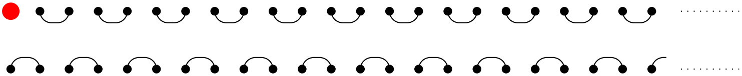
- $\alpha\beta \notin \mathcal{B}_\mathbb{N}$!
- So $\mathcal{B}_\mathbb{N}$ is not a semigroup!

$\alpha =$

$\beta =$



$\alpha\beta =$



Infinite Brauer monoids DON'T EXIST!

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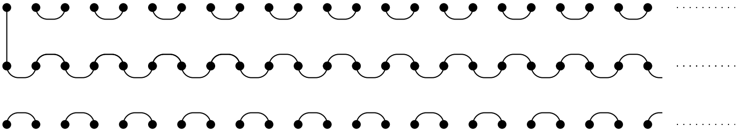

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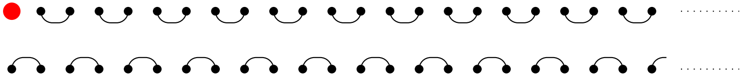

Infinite Brauer monoids DON'T EXIST!

- Consider $\alpha, \beta \in \mathcal{B}_{\mathbb{N}}$ below.
- $\alpha\beta \notin \mathcal{B}_{\mathbb{N}}$!
- So $\mathcal{B}_{\mathbb{N}}$ is not a semigroup!
- But $\alpha\beta \in \mathcal{PB}_{\mathbb{N}}$, the partial Brauer monoid.

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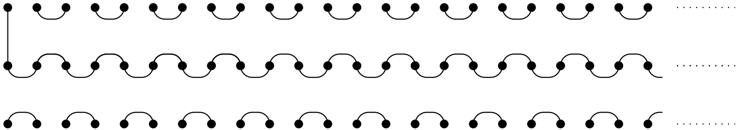
$\beta =$



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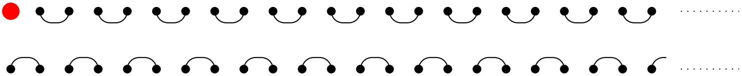

Partial Brauer monoids

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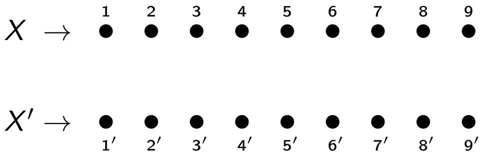

Partial Brauer monoids

- Let X be a set.
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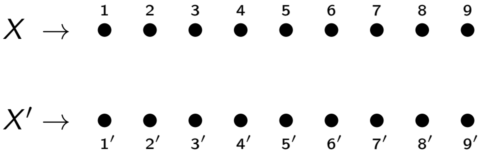

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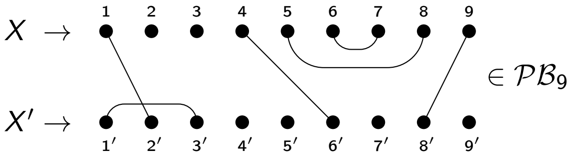

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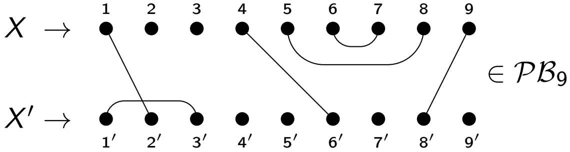
  $= \text{the partial Brauer monoid over } X$.



Partial Brauer monoids — product in $\mathcal{PB}_X$

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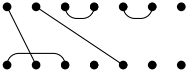
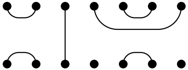
Let $\alpha, \beta \in \mathcal{PB}_X$.

$\alpha =$ 

$\beta =$ 

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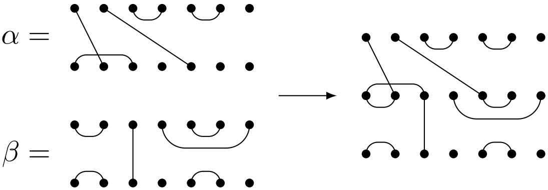

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Let $\alpha, \beta \in \mathcal{PB}_X$. To calculate $\alpha\beta$:

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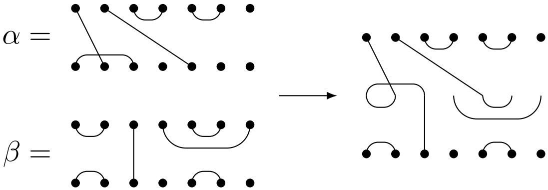

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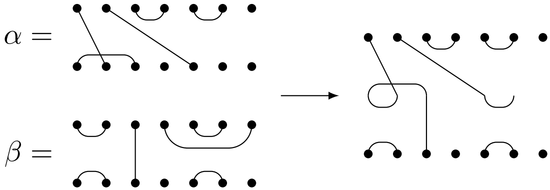

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Let $\alpha, \beta \in \mathcal{PB}_X$. To calculate $\alpha\beta$:

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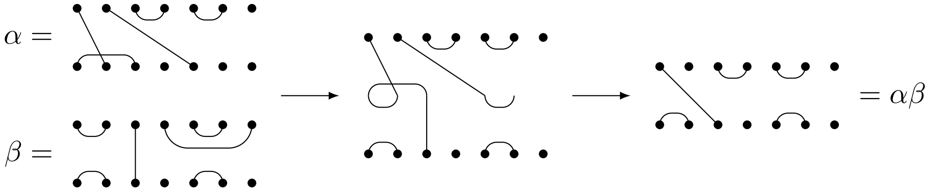
(3) smooth out (and prune) resulting graph to obtain $\alpha\beta$.



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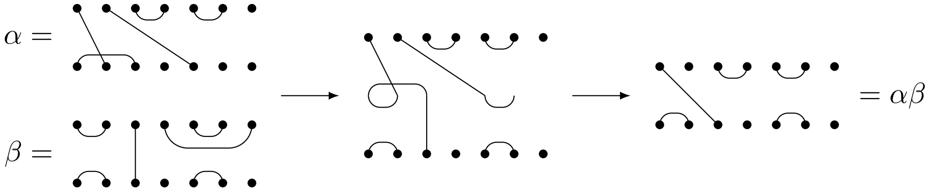

- The operation is associative, so $\mathcal{PB}_X$ is a semigroup (monoid, etc).

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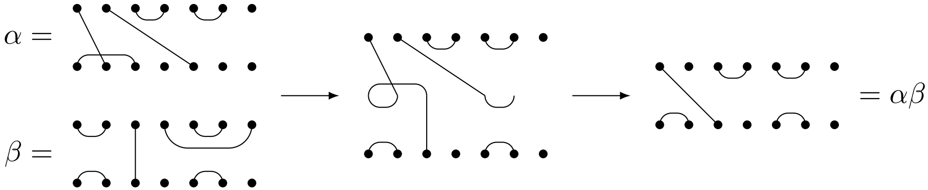

- The operation is associative, so $\mathcal{PB}_X$ is a semigroup (monoid, etc).
- No problems with infinite X.

Partial Brauer monoids — units and idempotents

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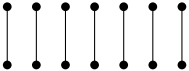
- $\mathcal{PB}_X$ has an identity element 1.

$1 =$ 

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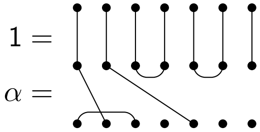

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- $\mathcal{PB}_X$ has an identity element 1.
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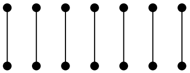
$1 =$ 

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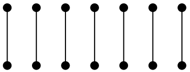
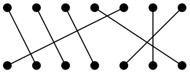
$1 =$ 

$\beta =$ 

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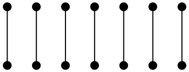

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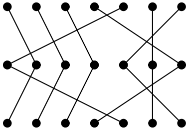

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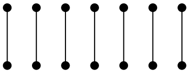
$1 =$ 

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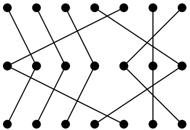
$\beta^* =$ 

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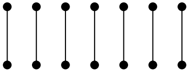
$1 =$ 

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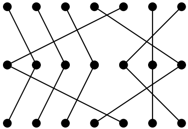

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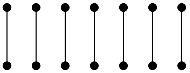

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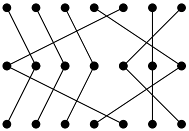

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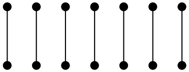
$1 =$ 

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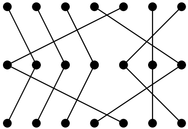
$\beta^* =$ 

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Next few pages:

- Idempotents and one-sided units in infinite partial Brauer monoids
  - J. Algebra **534** (2019) 427–482

Partial Brauer monoids — products of idempotents

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Theorem (inspired by Howie 1966)

Let $\mathbb{E}(\mathcal{PB}_X) = \langle E(\mathcal{PB}_X)\rangle$. Then

$$\mathbb{E}(\mathcal{PB}_X) = \big\{\alpha \in \mathcal{PB}_X : \mathrm{def}(\alpha) \leq 1 \text{ and } \mathrm{sh}(\alpha) = 0\big\}$$

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- $\mathrm{def}(\alpha) = |X \setminus \mathrm{dom}(\alpha)|$

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Theorem (inspired by Fountin and Lewin 1993)

Let $\mathbb{F}(\mathcal{PB}_X) = \big\langle E(\mathcal{PB}_X) \cup \mathbb{G}(\mathcal{PB}_X) \big\rangle$. Then

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Partial Brauer monoids — relative rank

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Theorem (inspired by Higgins, Howie, Ruškuc 1998)

- For $\alpha, \beta \in \mathcal{PB}_X$ as below, $\mathcal{PB}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$.

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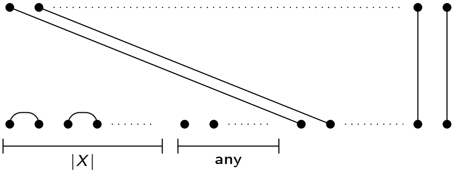

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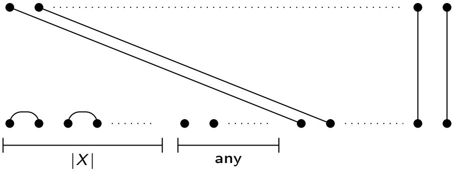
$\alpha =$ 

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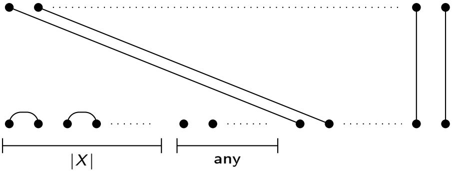

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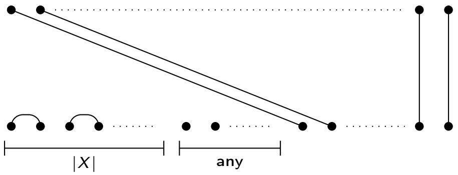

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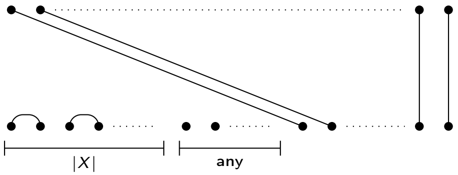

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- For $\alpha, \beta \in \mathcal{PB}_X$ as below, $\mathcal{PB}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$.
- In fact, $\mathcal{PB}_X = \alpha \mathcal{S}_X \beta$.
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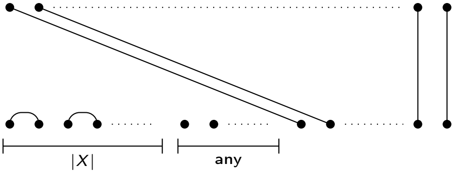
- $\mathcal{PB}_X = \langle \mathcal{B}_X \rangle = \mathcal{B}_X^2$.

$\alpha =$ 

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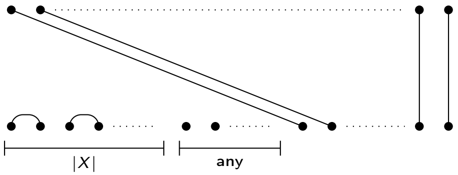
$\alpha =$ 

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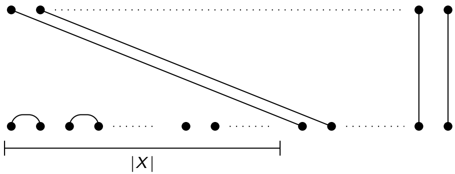
$\alpha =$ 

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Partial Brauer monoids — one-sided units

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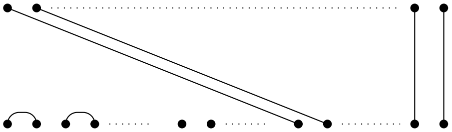
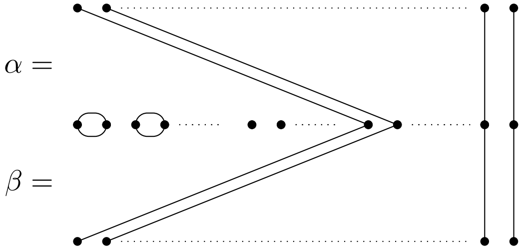
$\alpha =$ 

Partial Brauer monoids — one-sided units

- Consider α from the theorem(s).
- Then $\alpha\beta = 1$.



Partial Brauer monoids — one-sided units

- Consider α from the theorem(s).
- Then $\alpha\beta = 1$... but $\beta\alpha \neq 1$.

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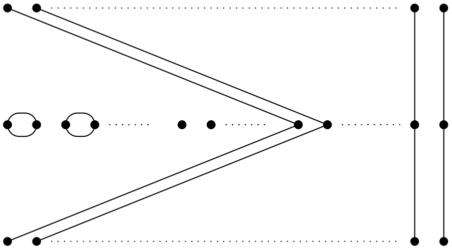
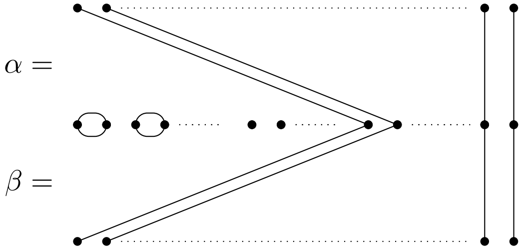


Partial Brauer monoids — one-sided units

- Consider α from the theorem(s).
- Then $\alpha\beta = 1$... but $\beta\alpha \neq 1$. So $\langle \alpha, \beta \rangle$ is bicyclic.



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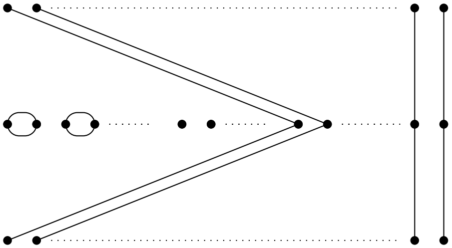

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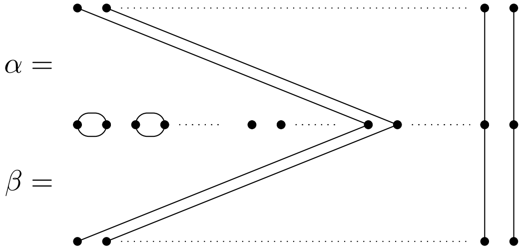

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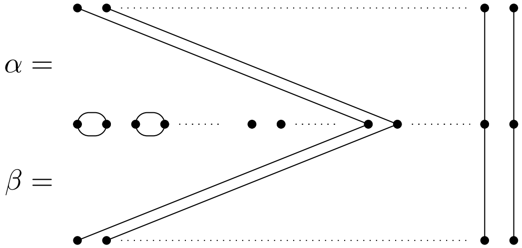

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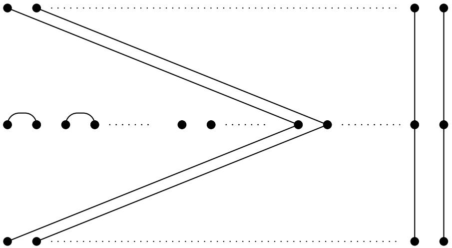

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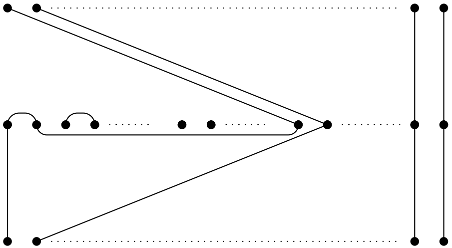

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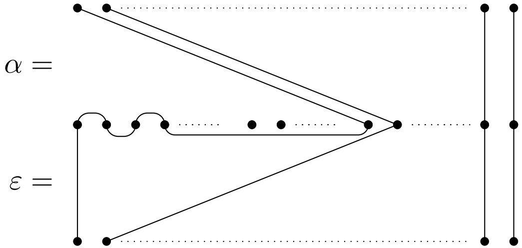

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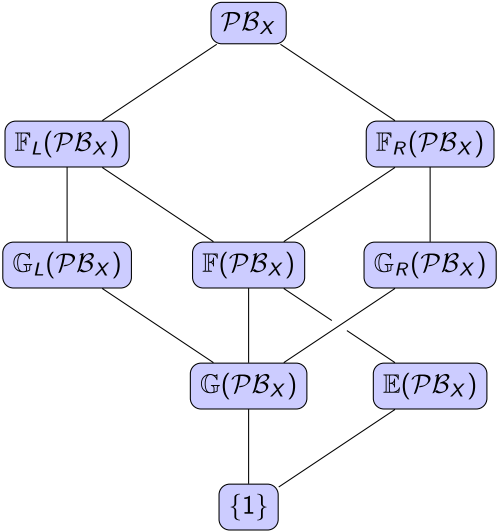
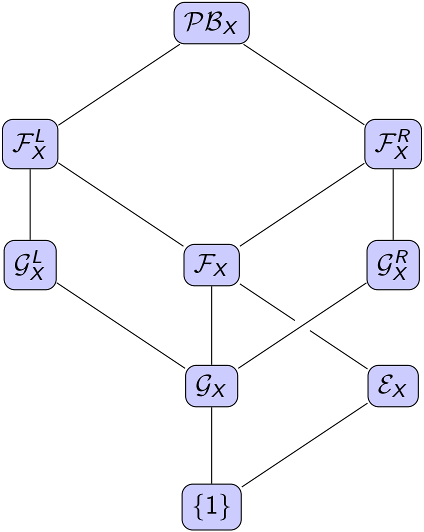
  where ρ is the number of infinite cardinals $\aleph_0 \leq \mu \leq |X|$.
- Generators modulo these submonoids are classified.

Partial Brauer monoids — submonoids



- $\mathbb{F}_L(\mathcal{PB}_X) = \langle E(\mathcal{PB}_X) \cup \mathbb{G}_L(\mathcal{PB}_X)\rangle$, etc.

Partial Brauer monoids — submonoids



- $\mathcal{F}_X^L = \mathbb{F}_L(\mathcal{PB}_X)$, etc.

Partial Brauer monoids — submonoids

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Idempotents and one-sided units in infinite partial Brauer monoids

James East

Centre for Research in Mathematics; School of Computing, Engineering and Mathematics, Western Sydney University, Locked Bag 1797, Penrith, NSW 2751, Australia

Lemma 4.1	Description of $\mathcal{G}_X^L$ and $\mathcal{G}_X$
Theorem 5.8	Description of $\mathcal{E}_X$
Theorem 6.1	Description of $\mathcal{F}_X$
Theorem 6.6	Description of $\mathcal{F}_X^L$
Theorem 4.7	$\operatorname{rank}(\mathcal{PB}_X : \mathcal{G}_X) = 2$
Theorem 4.9	$\operatorname{rank}(\mathcal{PB}_X : \mathcal{G}_X^L) = 1$
Theorem 5.12	$\operatorname{rank}(\mathcal{PB}_X : \mathcal{E}_X) = 2$
Theorem 6.3	$\operatorname{rank}(\mathcal{PB}_X : \mathcal{F}_X) = 2$
Theorem 7.1	$\operatorname{rank}(\mathcal{PB}_X : \mathcal{F}_X^L) = 1$

Theorem 7.6	$\operatorname{rank}(\mathcal{F}_X^L : \mathcal{F}_X) = 1 + \rho$
Theorem 7.7	$\operatorname{rank}(\mathcal{F}_X^L : \mathcal{E}_X) = 2^{\lvert X \rvert}$
Theorem 7.14	$\operatorname{rank}(\mathcal{F}_X^L : \mathcal{G}_X^L) = 2 + 2\rho$
Theorem 7.17	$\operatorname{rank}(\mathcal{F}_X^L : \mathcal{G}_X) = 3 + 3\rho$
Theorem 6.5	$\operatorname{rank}(\mathcal{F}_X : \mathcal{E}_X) = 2^{\lvert X \rvert}$
Theorem 6.16	$\operatorname{rank}(\mathcal{F}_X : \mathcal{G}_X) = 2 + 2\rho$
Theorem 4.12	$\operatorname{rank}(\mathcal{G}_X^L : \mathcal{G}_X) = 2 + 2\rho$
Theorem 8.3	Bergman/Sierpiński in $\mathcal{PB}_X$
Theorem 8.8	Bergman/Sierpiński in all other monoids

Partial Brauer monoids —Sierpiński and Bergman

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Theorem (inspired by Maltcev, Mitchell and Ruškuc)

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Theorem (inspired by Hyde and Péresse)

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- None of $\mathcal{E}_X, \mathcal{G}_X^L, \mathcal{G}_X^R, \mathcal{F}_X, \mathcal{F}_X^L, \mathcal{F}_X^R$ have the Bergman property.

Monoids

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- $\mathbb{G}_R(M) = \{\text{right units of } M\}$
- $\mathbb{F}(M) = \langle E(M) \cup \mathbb{G}(M) \rangle$

Monoids

For a monoid M, let

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- $\mathbb{G}(M) = \{\text{units of } M\}$
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- $\mathbb{F}_R(M) = \langle E(M) \cup \mathbb{G}_R(M) \rangle$
- $\mathbb{F}_{LR}(M) = \langle E(M) \cup \mathbb{G}_L(M) \cup \mathbb{G}_R(M) \rangle$
- $\mathbb{I}(M) = M$

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- $\mathbb{I}(M) = M$
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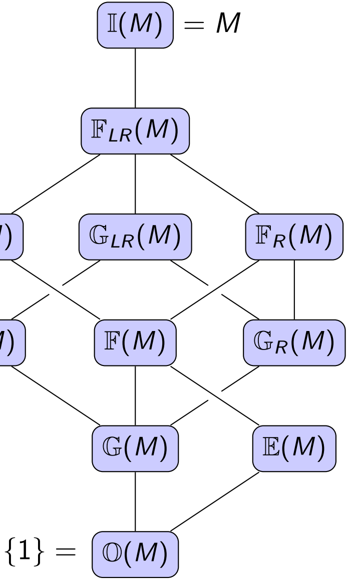
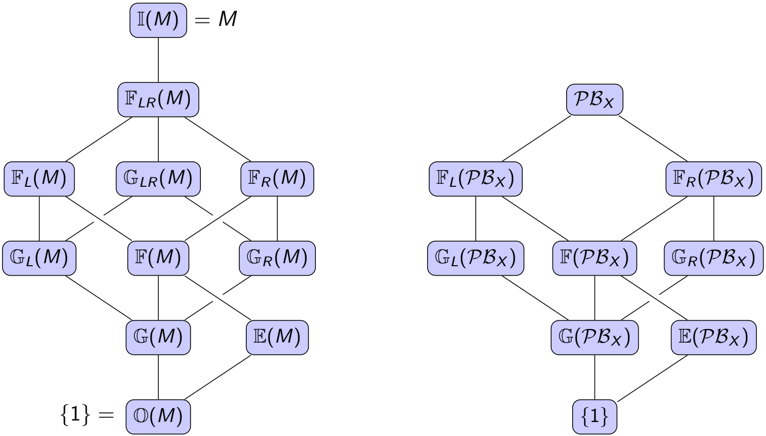
- $\mathbb{F}_{LR}(M) = \langle E(M) \cup \mathbb{G}_L(M) \cup \mathbb{G}_R(M) \rangle$
- $\mathbb{I}(M) = M$
- $\mathbb{O}(M) = \{1\}$
- All are submonoids of M.

Submonoids

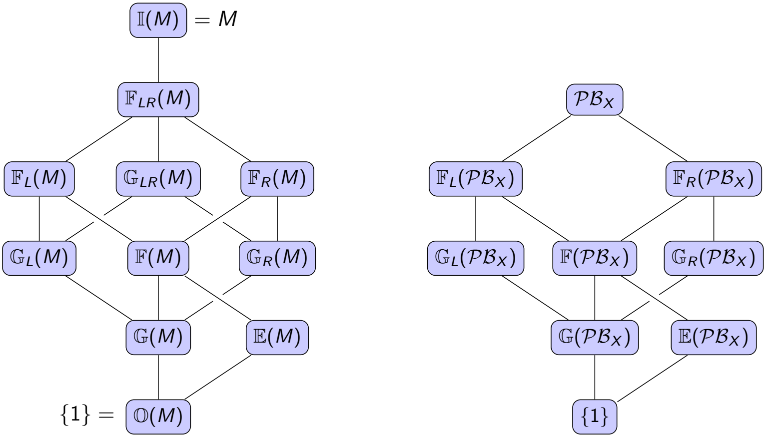


Submonoids



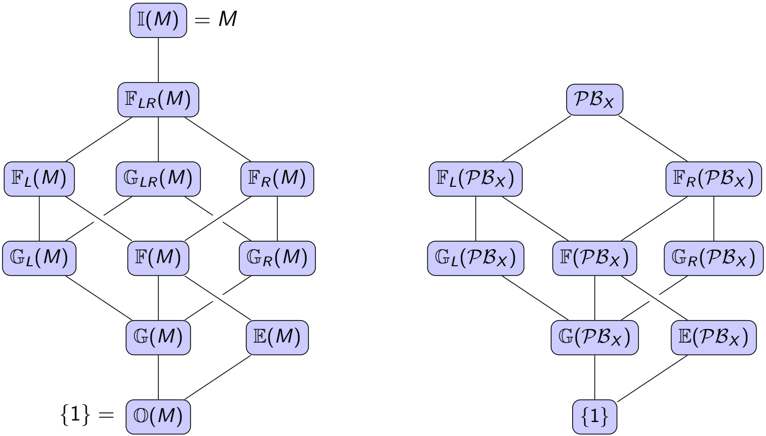
$\mathbb{I}(M) = M$
$\mathbb{F}_{LR}(M)$
$\mathbb{F}_L(M)$
$\mathbb{G}_{LR}(M)$
$\mathbb{F}_R(M)$
$\mathbb{G}_L(M)$
$\mathbb{F}(M)$
$\mathbb{G}_R(M)$
$\mathbb{G}(M)$
$\mathbb{E}(M)$
$\{1\} = \mathbb{O}(M)$
$\mathcal{PB}_X$
$\mathbb{F}_L(\mathcal{PB}_X)$
$\mathbb{F}_R(\mathcal{PB}_X)$
$\mathbb{G}_L(\mathcal{PB}_X)$
$\mathbb{F}(\mathcal{PB}_X)$
$\mathbb{G}_R(\mathcal{PB}_X)$
$\mathbb{G}(\mathcal{PB}_X)$
$\mathbb{E}(\mathcal{PB}_X)$
$\{1\}$

Submonoids



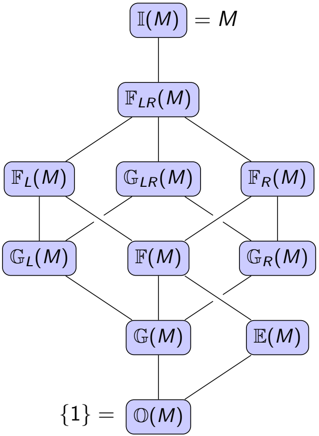
- WT$\mathbb{F}_{LR}$?

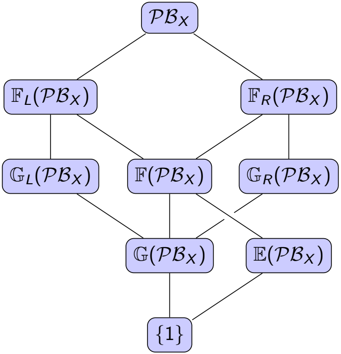
Submonoids



- WT$\mathbb{F}_{LR}$?
- Earlier theorem: $\mathcal{PB}_X = \langle \mathbb{G}_L(\mathcal{PB}_X) \cup \mathbb{G}_R(\mathcal{PB}_X) \rangle$.

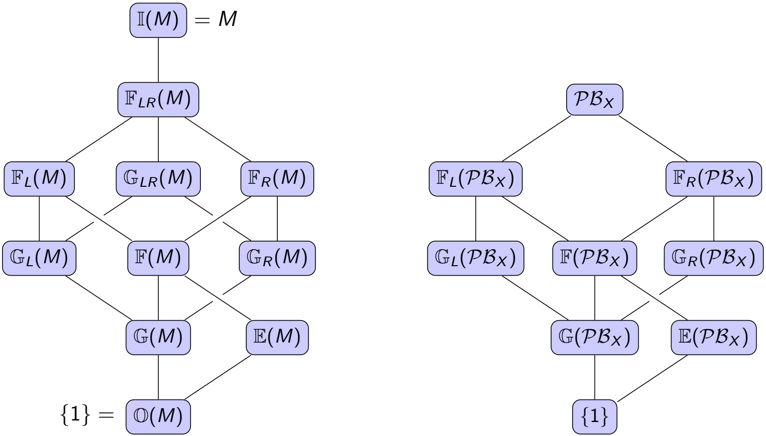
Submonoids





- WT$\mathbb{F}_{LR}$?
- Earlier theorem: $\mathcal{PB}_X = \mathbb{G}_{LR}(\mathcal{PB}_X)$

Submonoids



- WT$\mathbb{F}_{LR}$?
- Earlier theorem: $\mathcal{PB}_X = \mathbb{G}_{LR}(\mathcal{PB}_X) = \mathbb{F}_{LR}(\mathcal{PB}_X)$!

Functors

- Let $\mathcal{M}$ be the (monoidal) category of monoids.

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- So are all the rest.

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- Quiz: $\mathbb{G}_L(\mathbb{G}_L(M)) =$

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- Quiz: $\mathbb{G}_L(\mathbb{G}_L(M)) = \mathbb{G}(M)$! ...... $\mathbb{G}_L \circ \mathbb{G}_L = \mathbb{G}$.
- Quiz: $\mathbb{E}(\mathbb{E}(M)) = \mathbb{E}(M)$!

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- Quiz: $\mathbb{G}_L(\mathbb{G}_L(M)) = \mathbb{G}(M)$! ...... $\mathbb{G}_L \circ \mathbb{G}_L = \mathbb{G}$.
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- So are all the rest.
- Quiz: $\mathbb{G}_L(\mathbb{G}_L(M)) = \mathbb{G}(M)$! ...... $\mathbb{G}_L \circ \mathbb{G}_L = \mathbb{G}$.
- Quiz: $\mathbb{E}(\mathbb{E}(M)) = \mathbb{E}(M)$! ...... $\mathbb{E} \circ \mathbb{E} = \mathbb{E}$.
- $\mathbb{E}(\mathbb{G}(M))$

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- So are all the rest.
- Quiz: $\mathbb{G}_L(\mathbb{G}_L(M)) = \mathbb{G}(M)$! ...... $\mathbb{G}_L \circ \mathbb{G}_L = \mathbb{G}$.
- Quiz: $\mathbb{E}(\mathbb{E}(M)) = \mathbb{E}(M)$! ...... $\mathbb{E} \circ \mathbb{E} = \mathbb{E}$.
- $\mathbb{E}(\mathbb{G}(M)) = \{1\}$

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- Quiz: $\mathbb{E}(\mathbb{E}(M)) = \mathbb{E}(M)$! ...... $\mathbb{E} \circ \mathbb{E} = \mathbb{E}$.
- $\mathbb{E}(\mathbb{G}(M)) = \{1\} = \mathbb{G}(\mathbb{E}(M))$.

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- Quiz: $\mathbb{E}(\mathbb{E}(M)) = \mathbb{E}(M)$! ...... $\mathbb{E} \circ \mathbb{E} = \mathbb{E}$.
- $\mathbb{E}(\mathbb{G}(M)) = \{1\} = \mathbb{G}(\mathbb{E}(M))$. ...... $\mathbb{E} \circ \mathbb{G} = \mathbb{G} \circ \mathbb{E} = \mathbb{O}$.

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- Quiz: $\mathbb{E}(\mathbb{E}(M)) = \mathbb{E}(M)$! ...... $\mathbb{E} \circ \mathbb{E} = \mathbb{E}$.
- $\mathbb{E}(\mathbb{G}(M)) = \{1\} = \mathbb{G}(\mathbb{E}(M))$. ...... $\mathbb{E} \circ \mathbb{G} = \mathbb{G} \circ \mathbb{E} = \mathbb{O}$.
- $\mathbb{X} \circ \mathbb{I} = \mathbb{X} = \mathbb{I} \circ \mathbb{X}$

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- $\mathbb{X} \circ \mathbb{I} = \mathbb{X} = \mathbb{I} \circ \mathbb{X}$ and $\mathbb{X} \circ \mathbb{O} = \mathbb{O} = \mathbb{O} \circ \mathbb{X}$ for any $\mathbb{X}$.

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- So we have a monoid of functors,

$$\mathscr{F} = \{\mathbb{O}, \mathbb{E}, \mathbb{G}, \mathbb{G}_L, \mathbb{G}_R, \mathbb{G}_{LR}, \mathbb{F}, \mathbb{F}_L, \mathbb{F}_R, \mathbb{F}_{LR}, \mathbb{I}\}...$$

Functors

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- So are all the rest.
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- $\mathbb{E}(\mathbb{G}(M)) = \{1\} = \mathbb{G}(\mathbb{E}(M))$. ...... $\mathbb{E} \circ \mathbb{G} = \mathbb{G} \circ \mathbb{E} = \mathbb{O}$.
- $\mathbb{X} \circ \mathbb{I} = \mathbb{X} = \mathbb{I} \circ \mathbb{X}$ and $\mathbb{X} \circ \mathbb{O} = \mathbb{O} = \mathbb{O} \circ \mathbb{X}$ for any $\mathbb{X}$.

- So we have a monoid of functors,

$$\mathscr{F} = \{\mathbb{O}, \mathbb{E}, \mathbb{G}, \mathbb{G}_L, \mathbb{G}_R, \mathbb{G}_{LR}, \mathbb{F}, \mathbb{F}_L, \mathbb{F}_R, \mathbb{F}_{LR}, \mathbb{I}\}$$ .... right?

Composing functors

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$		$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$		$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$		$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$		$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{F}_{LR}$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$

Composing functors

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{F}_{LR}$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$

Composing functors

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{F}_{LR}$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$

- Are these really new functors?

Composing functors

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{F}_{LR}$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$

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- *Now* do we have a monoid of functors,

$$\mathscr{F} = \{\mathbb{O}, \mathbb{E}, \mathbb{G}, \mathbb{G}_L, \mathbb{G}_R, \mathbb{G}_{LR}, \mathbb{F}, \mathbb{F}_L, \mathbb{F}_R, \mathbb{F}_{LR}, \mathbb{Q}, \mathbb{P}, \mathbb{P}_L, \mathbb{P}_R, \mathbb{I}\}?$$

Composing functors

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{F}_{LR}$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$

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Composing functors

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{F}_{LR}$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{I}$

- Are these really new functors? .... Yes!
- *Now* do we have a monoid of functors, ... Yes!

$$\mathscr{F} = \{\mathbb{O}, \mathbb{E}, \mathbb{G}, \mathbb{G}_L, \mathbb{G}_R, \mathbb{G}_{LR}, \mathbb{F}, \mathbb{F}_L, \mathbb{F}_R, \mathbb{F}_{LR}, \mathbb{Q}, \mathbb{P}, \mathbb{P}_L, \mathbb{P}_R, \mathbb{I}\}?$$

The monoid $\mathscr{F}$

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}_L$	$\mathbb{P}_R$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_{LR}$
$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$
$\mathbb{P}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$
$\mathbb{P}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$
$\mathbb{P}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}_L$	$\mathbb{P}_R$	$\mathbb{I}$

The monoid $\mathscr{F}$

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}_L$	$\mathbb{P}_R$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_{LR}$
$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$
$\mathbb{P}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$
$\mathbb{P}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$
$\mathbb{P}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}_L$	$\mathbb{P}_R$	$\mathbb{I}$

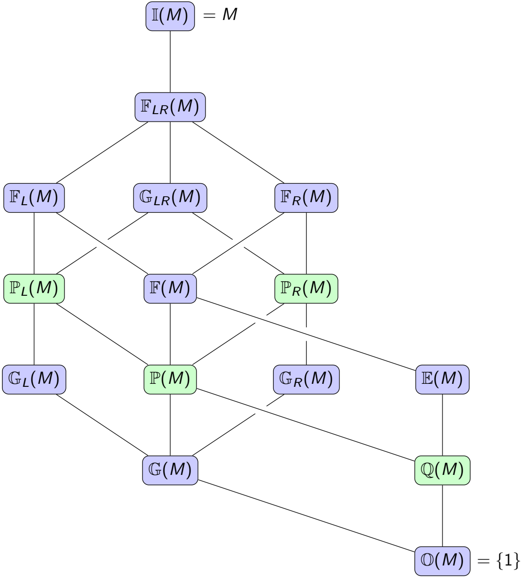
- So $\mathscr{F} = \{\mathbb{O}, \mathbb{E}, \mathbb{G}, \mathbb{G}_L, \ldots, \mathbb{I}\}$ is a monoid.

The monoid $\mathscr{F}$

$\circ$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}_L$	$\mathbb{P}_R$	$\mathbb{I}$
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$
$\mathbb{E}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{E}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{E}$
$\mathbb{G}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$
$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_L$
$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_R$
$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$
$\mathbb{F}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}$
$\mathbb{F}_L$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_L$
$\mathbb{F}_R$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_R$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_R$
$\mathbb{F}_{LR}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{F}_{LR}$
$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Q}$
$\mathbb{P}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}$
$\mathbb{P}_L$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_L$
$\mathbb{P}_R$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$	$\mathbb{O}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{G}$	$\mathbb{P}_R$
$\mathbb{I}$	$\mathbb{O}$	$\mathbb{E}$	$\mathbb{G}$	$\mathbb{G}_L$	$\mathbb{G}_R$	$\mathbb{G}_{LR}$	$\mathbb{F}$	$\mathbb{F}_L$	$\mathbb{F}_R$	$\mathbb{F}_{LR}$	$\mathbb{Q}$	$\mathbb{P}$	$\mathbb{P}_L$	$\mathbb{P}_R$	$\mathbb{I}$

- So $\mathscr{F} = \{\mathbb{O}, \mathbb{E}, \mathbb{G}, \mathbb{G}_L, \dots, \mathbb{I}\}$ is a monoid...... and $|\mathscr{F}| \leq 15$.

The size of $\mathscr{F}$



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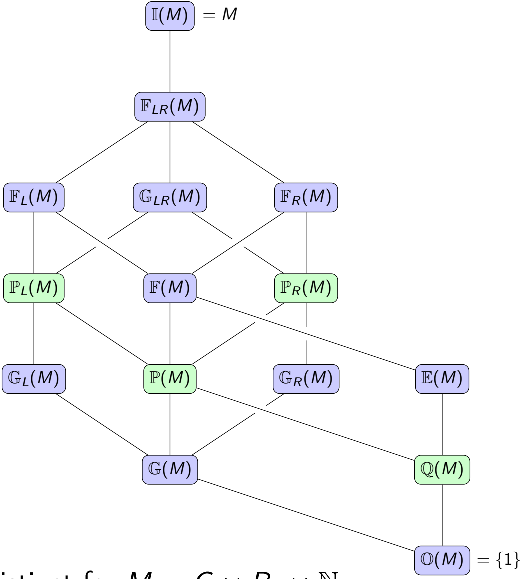

- The above are all distinct for $M = G \times B_0 \times \mathbb{N}$.

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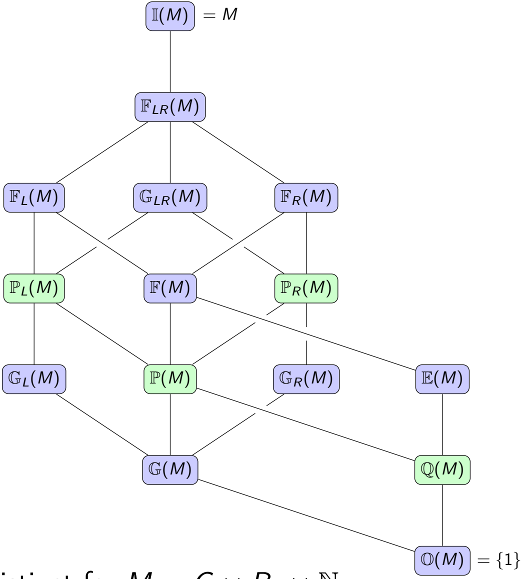

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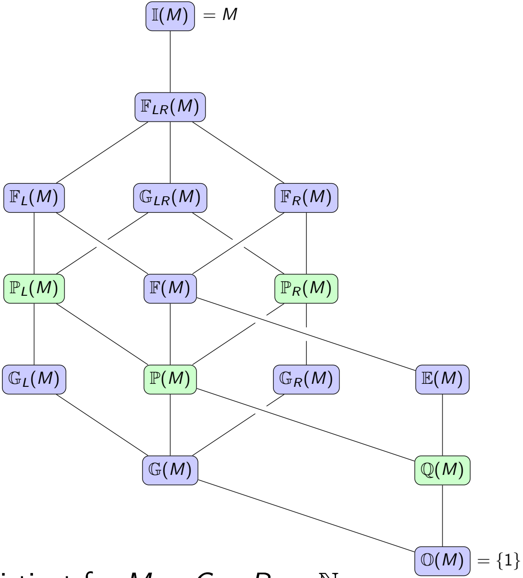

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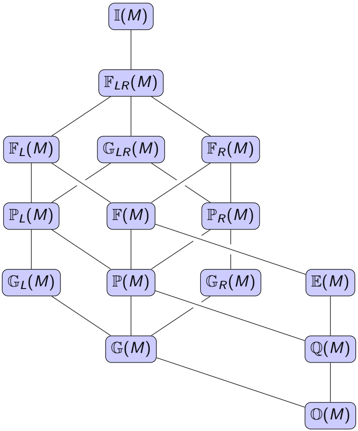
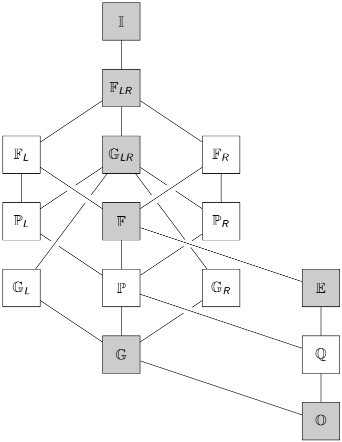
The structure of $\mathscr{F}$



$\mathscr{L}(M)$



$\mathscr{F}$

The lattice $\mathscr{L}(M)$

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- Observation: $\mathbb{G}_L(M) = \mathbb{G}(M) \Leftrightarrow \mathbb{G}_R(M) = \mathbb{G}(M)$.
- $\mathscr{L}(M)$ simplifies greatly for such M.

When $\mathbb{G}_L(M) = \mathbb{G}(M) = \mathbb{G}_R(M)$

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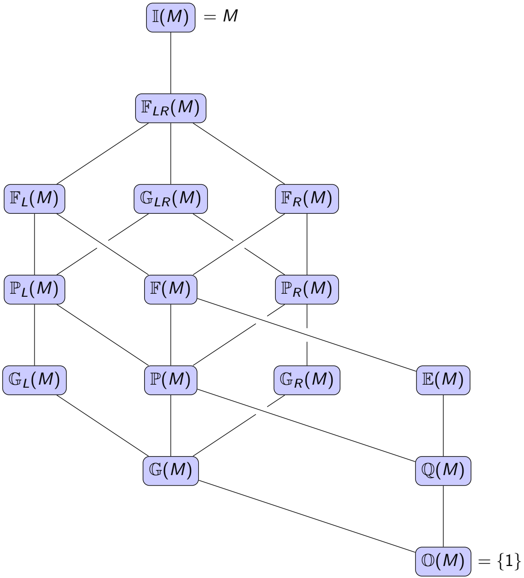

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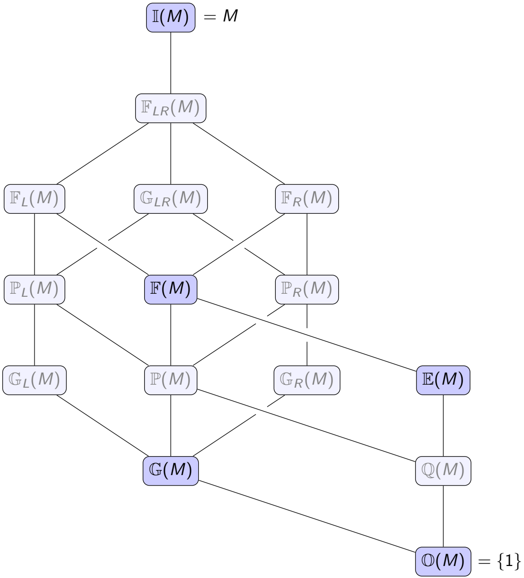

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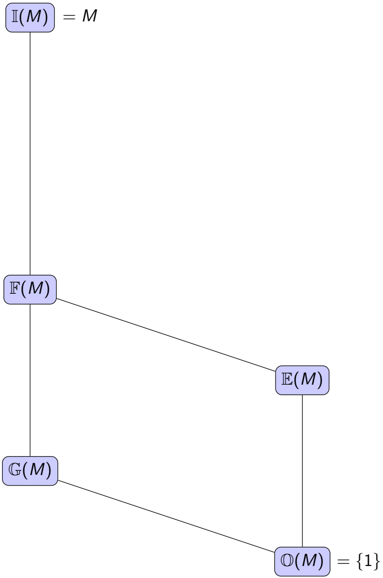

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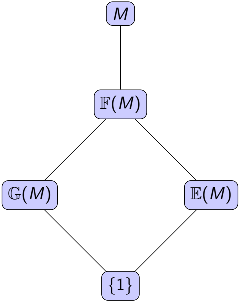

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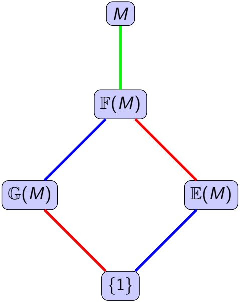

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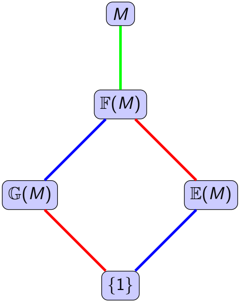

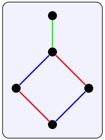

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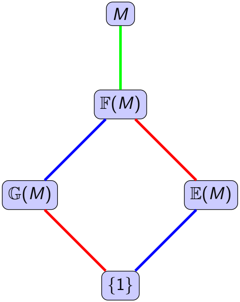

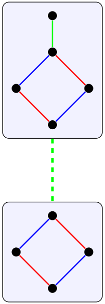

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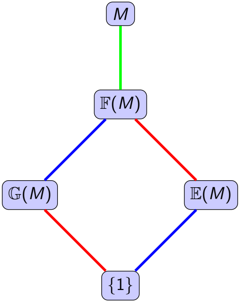

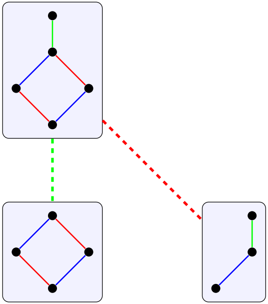

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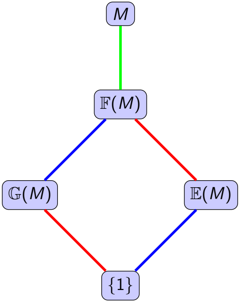

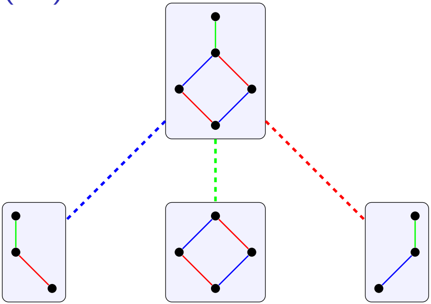

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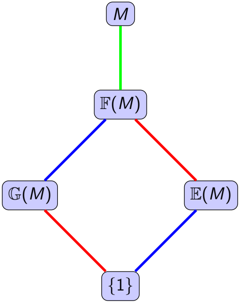

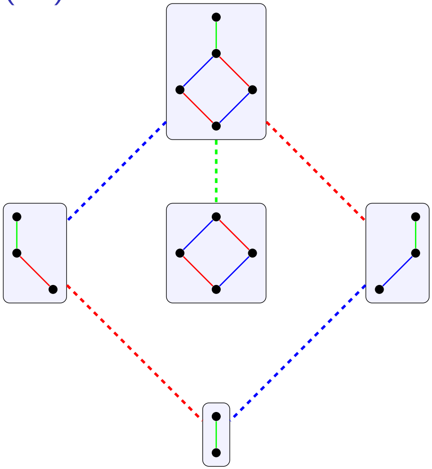

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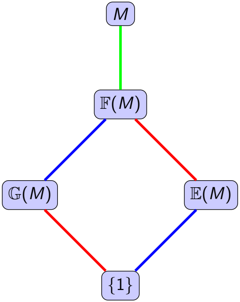

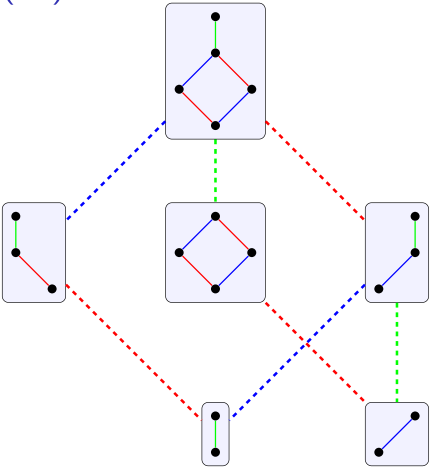

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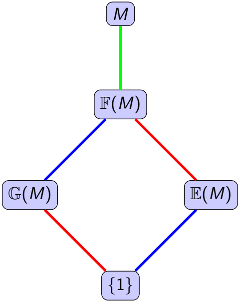

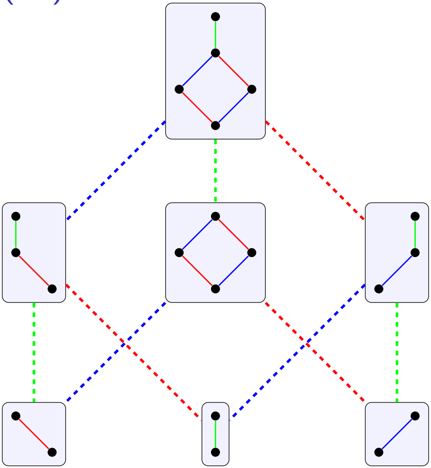

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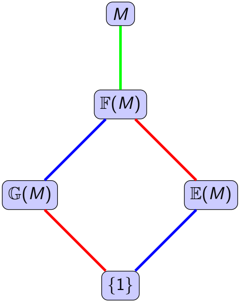

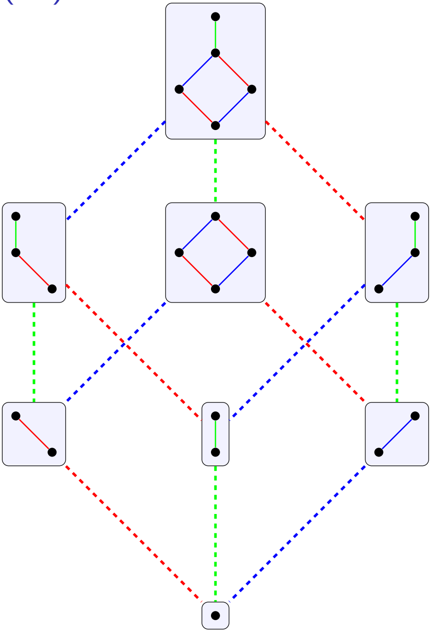

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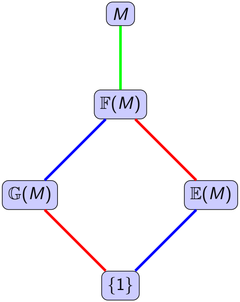

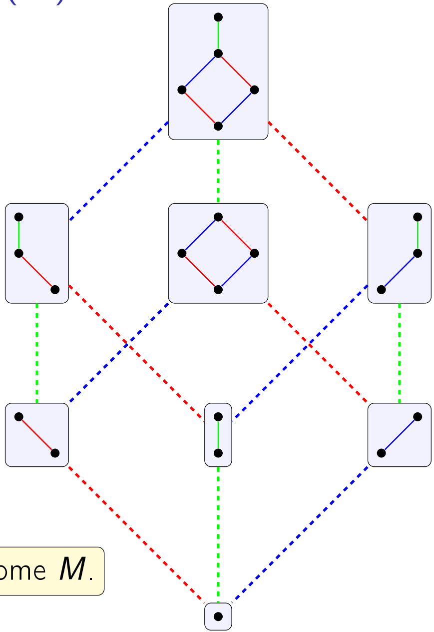

Theorem: all are realised by some M.

When $\mathbb{G}_L(M) \neq \mathbb{G}(M) \neq \mathbb{G}_R(M)$

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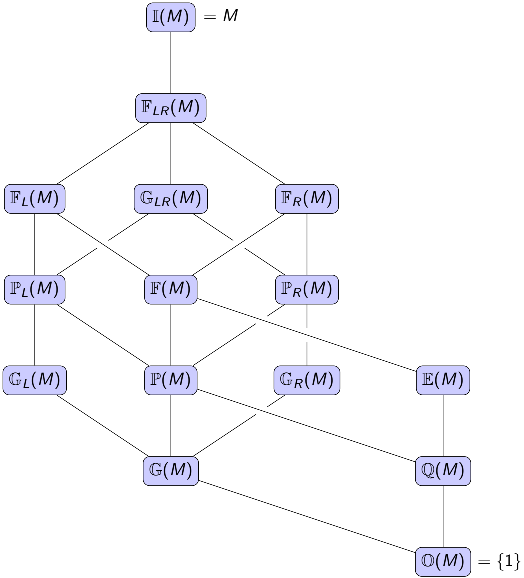

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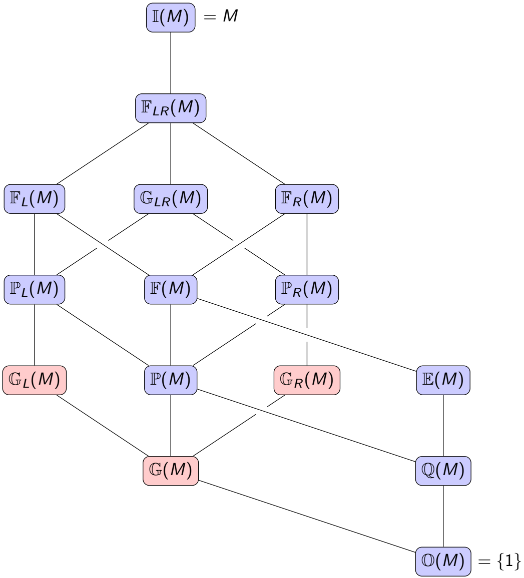

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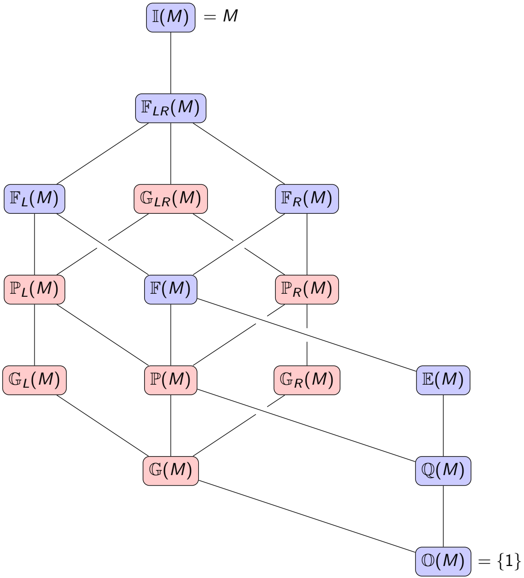

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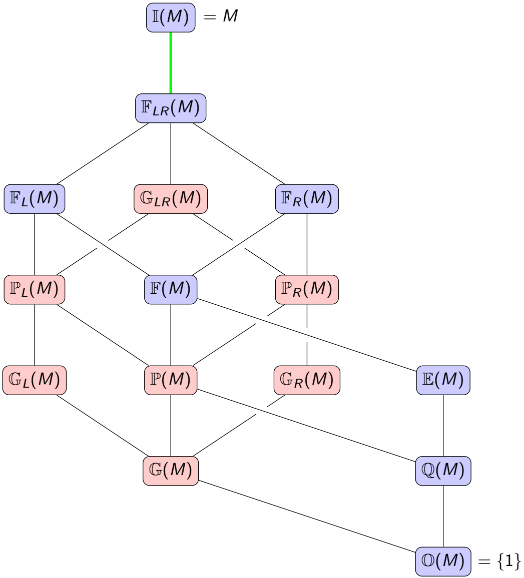

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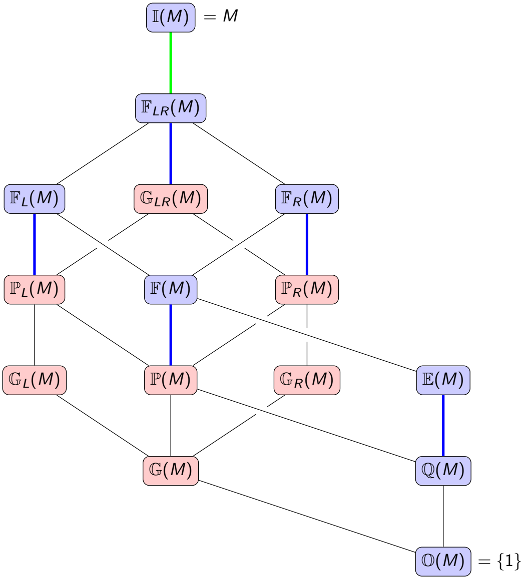

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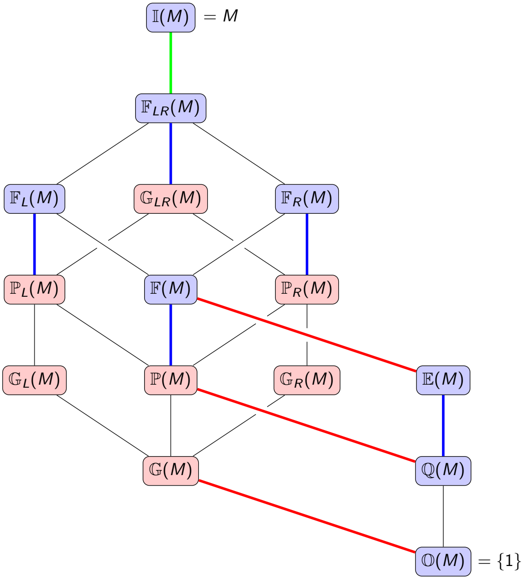

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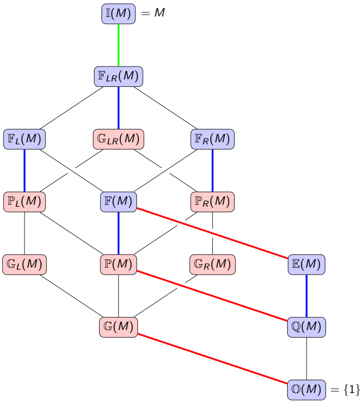

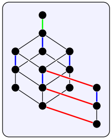

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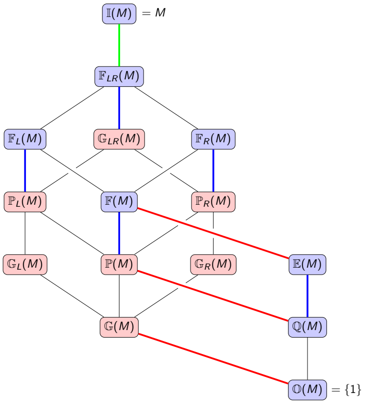

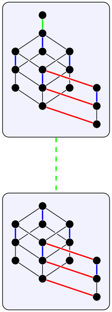

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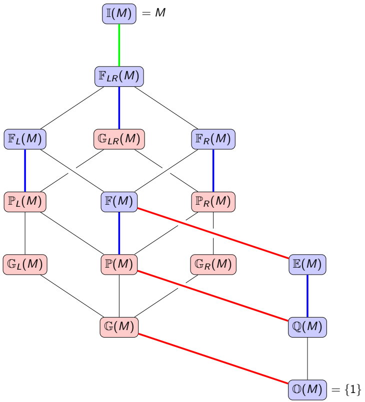

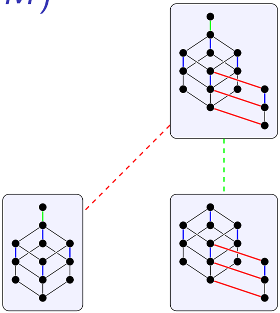

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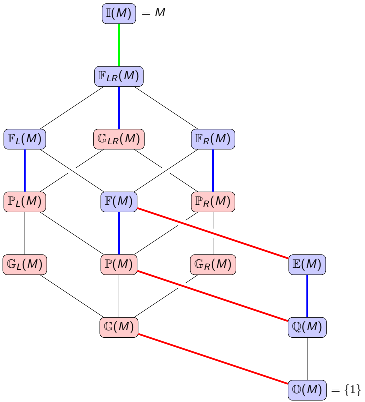

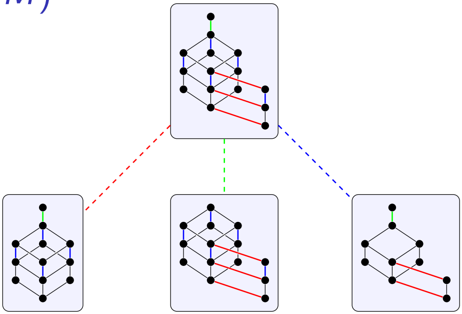

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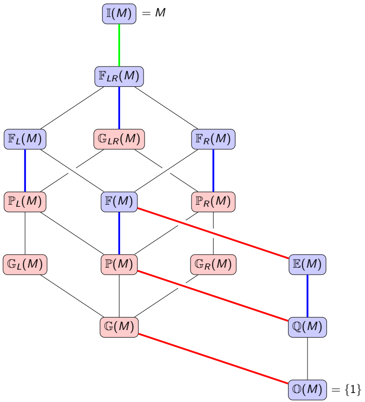

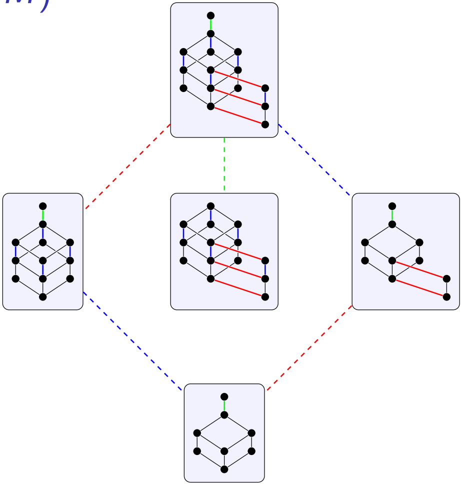

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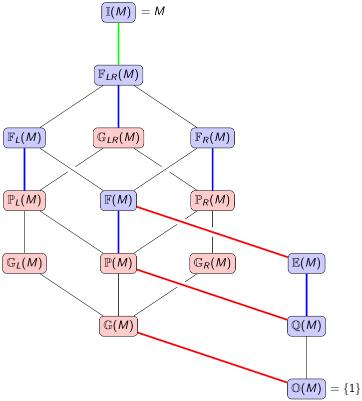

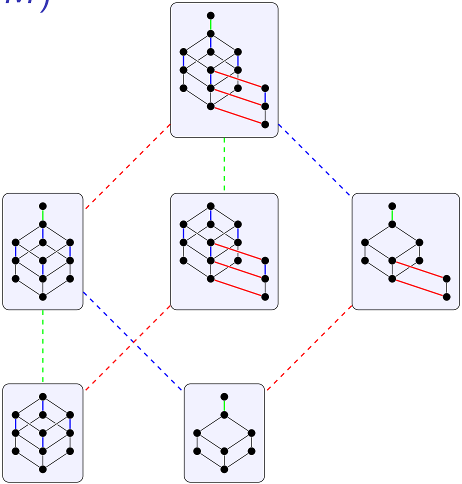

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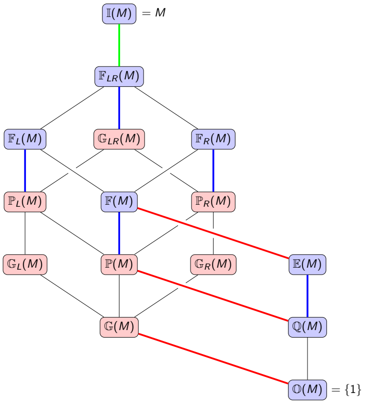

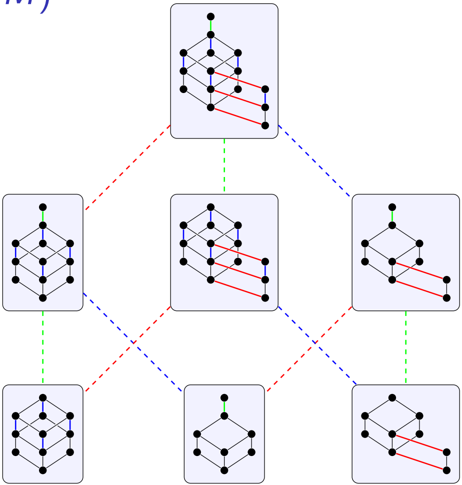

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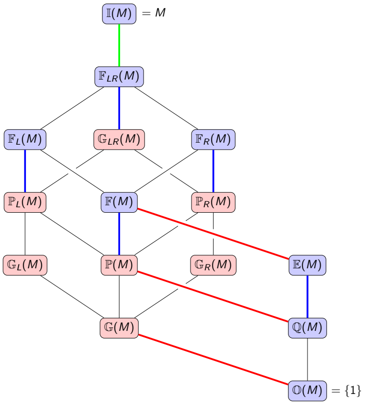

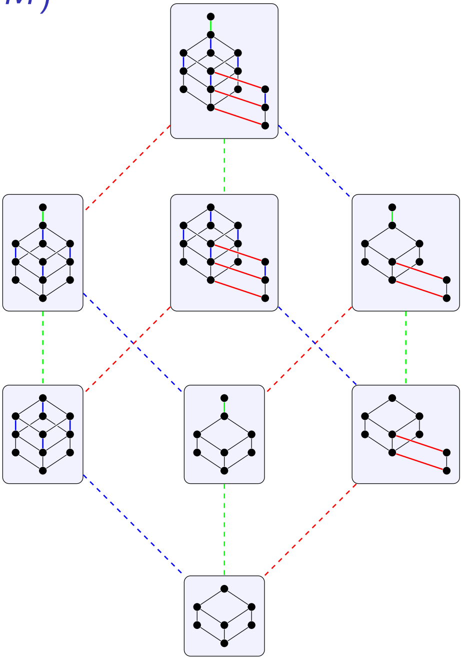

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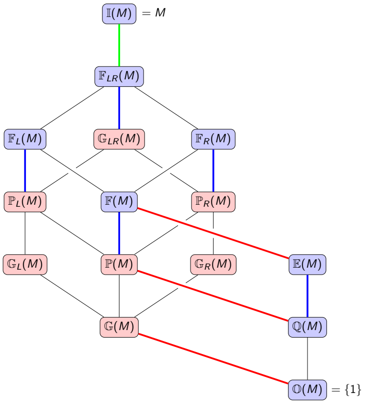
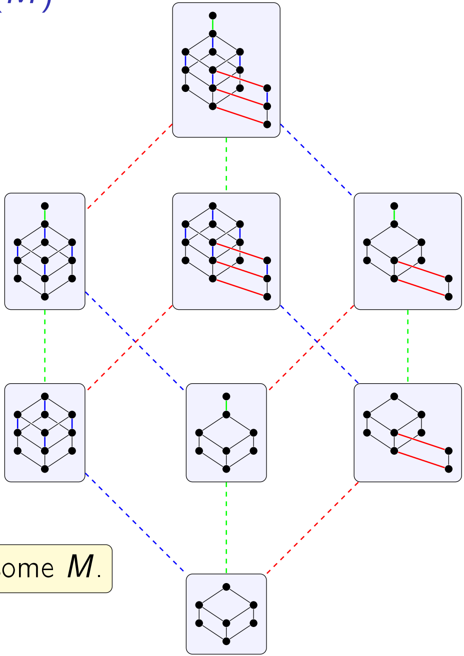


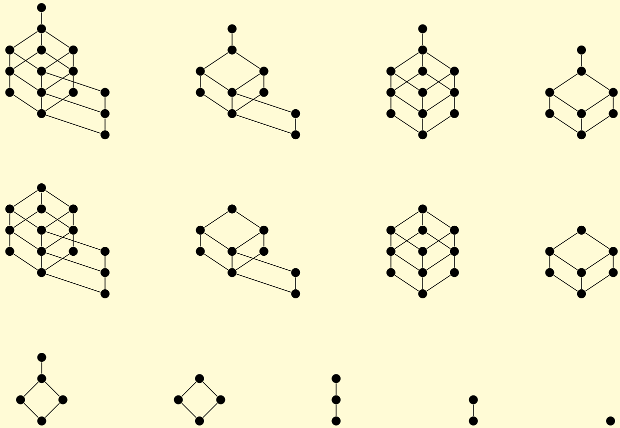


Theorem: all are realised by some M.

Classification of lattices

Theorem (inspired by the Old White Swan)

Up to isomorphism, the possible lattices $\mathscr{L}(M)$ are:



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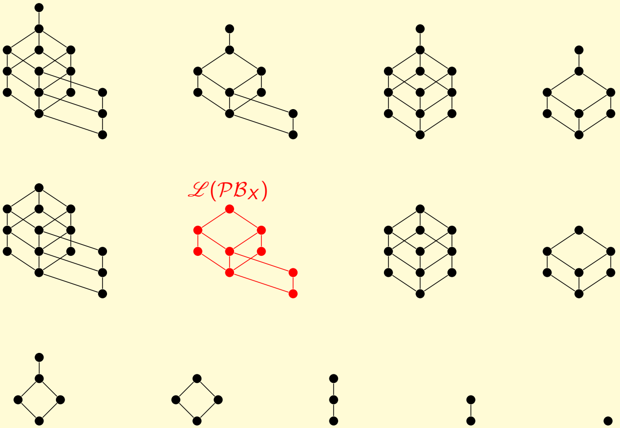

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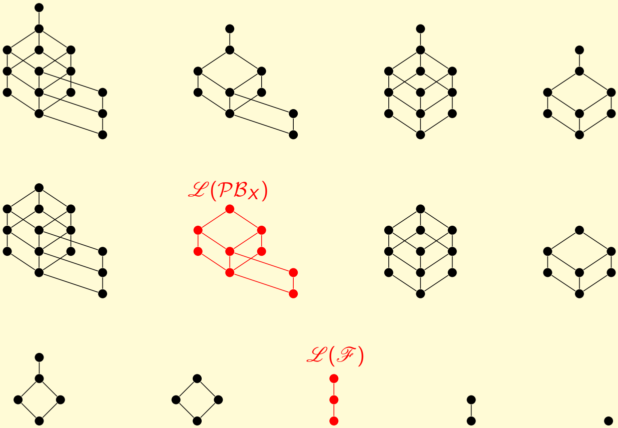

Thank you

- Idempotents and one-sided units in infinite partial Brauer monoids
  - J. Algebra **534** (2019) 427–482
- A semigroup of functors on the category of monoids
  - Coming soon...