The Howson property for one-sided ideals of a semigroup

Scott Carson

January 10, 2019

Motivation

In early 2017, Ben Steinberg contacted Vicky looking to find some interesting examples of semigroups of which the intersection of finitely generated right ideals are finitely generated. In particular, he was interested in such examples *besides* the ones where the intersection of principal right ideals are either principal or empty.

To which Vicky conjectured a number of examples. As for the details...

"**I could give all this to a student**"

Howson property

Definition

An algebra is said to have the **Howson property** (otherwise known as the **finitely generated intersection property (FGIP)**) if the intersection of any two finitely generated subalgebras is also finitely generated.

We define a corresponding notion of the Howson property for one-sided ideals of semigroups.

Definition

A semigroup S is called **right** (resp. **left**) **Howson** if the intersection of finitely generated right (left) ideals of S are finitely generated.

Note that a semigroup is right (resp. left) Howson if the intersection of principal right (left) ideals is finitely generated.

Ultra-Howson semigroups

Definition

A semigroup is **right** (resp. **left**) **ultra-Howson** if the intersection of any two principal right (left) ideals is finitely generated and there exists at least one such intersection that is not principal.

We then combine these definitions to form the two-sided case.

Definition

A semigroup is **ultra-Howson** if it is both left and right ultra-Howson.

Notation

For the purpose of this talk, it will be useful to include some notation for some special sets of ultra-Howson semigroups. In particular

- $\mathbf{UHow}_{\mathrm{R}}$ = set of all right ultra-Howson semigroups
- $\mathbf{UHow}_{\mathrm{L}}$ = set of all left ultra-Howson semigroups
- $\mathbf{UHow}$ = set of all ultra-Howson semigroups
- $\mathbf{CUHow}$ = set of all commutative ultra-Howson semigroups

Let us pause for a moment to consider some important classes of semigroups, and any ultra-Howson semigroups they may contain.

Non-examples: Howson semigroups that are not ultra-Howson

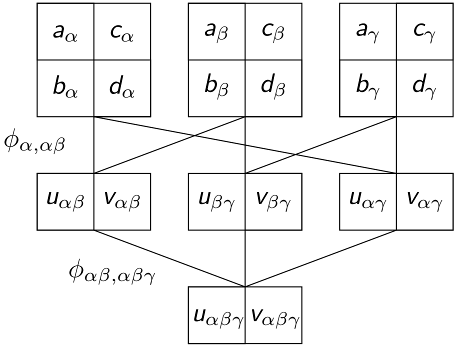
Of course, while many examples of semigroups we see exhibit the Howson property for one-sided principal ideals, most lack the property of being ultra-Howson.

groups	**inverse semigroups**
rectangular bands	**null semigroups**
left/right zero semigroups	**free semigroups**
free commutative semigroups	**. . .**

It all begs the question: do right or left ultra-Howson semigroups even exist? If so, what examples can we find?

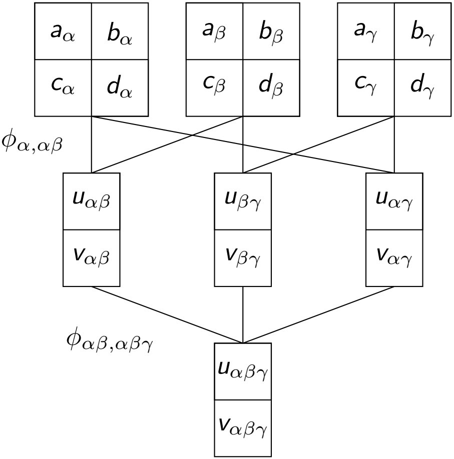
Example 1

This strong semilattice of rectangular bands is an example of a right ultra-Howson semigroup.



Example 2

Dually, this strong semilattice of rectangular bands is an example of a left ultra-Howson semigroup.



Motivating questions

It is clear then that examples of right and left ultra-Howson semigroups exist at least in the semigroup variety of normal bands. However, there are still questions left unanswered.

- Do any **'canonical' examples** of semigroups of this kind exist?
- Is it possible to **determine semigroup varieties** that contain examples of these semigroups?
- Given an ultra-Howson semigroup, it is possible to **construct** more examples?

Attempting to answer these (seemingly simple) questions formed the basis for our project.

Canonical example 1

Fix $n \in \mathbb{N}$ where $n \geq 2$ and let $\mathbf{n} = \{1, \ldots, n\}$. Let $\mathscr{A}$ denote the alphabet

$$\mathscr{A} = \{a, b, u_i, v_i : i \in \mathbf{n}\}.$$

Define ρ and λ to be relations on $\mathscr{A}^+$ given by

$$\rho = \{(au_i, bv_i) : i \in \mathbf{n}\} \quad \lambda = \{(u_i a, v_i b) : i \in \mathbf{n}\}.$$

Using the convention that $S_\alpha = \langle \mathscr{A} : \alpha \rangle$, we proved the following proposition.

(SC,VG) Proposition 1

The semigroup $S_\rho \in \mathbf{UHow}_{\mathsf{R}}$ and dually $S_\lambda \in \mathbf{UHow}_{\mathsf{L}}$.

In addition, we were able to show that both S_ρ and S_λ are cancellative semigroups.

Canonical example 2

By fixing $n \in \mathbb{N}$ where $n \geq 2$ and considering $\mathscr{A}$ as before, we define the following relation on $\mathscr{A}^+$

$$\gamma = \rho \cup \{(xy, yx) : x, y \in \mathscr{A}\}.$$

We proceeded to prove the following proposition.

(SC,VG) Proposition 2

The semigroup $S_\gamma \in$ **CUHow**.

However, this is not a cancellative semigroup. For example, where $i \neq j \in \mathbf{n}$

$$(au_i)v_j \sim_{\gamma^\sharp} (bv_i)v_j \sim_{\gamma^\sharp} (bv_j)v_i \sim_{\gamma^\sharp} (au_j)v_i$$

then we notice that

$$u_i v_j \nsim_{\gamma^\sharp} u_j v_i.$$

Canonical example 3

By fixing $n \in \mathbb{N}$ where $n \geq 2$ and considering $\mathscr{A}$ as before, we then defined the following relation on $\mathscr{A}^+$

$$\eta = \gamma \cup \{(u_i v_j, u_j v_i) : i, j \in \mathbf{n}\}.$$

By adjusting the approach we used to obtain Proposition 2, we showed the following.

(SC,VG) Proposition 3

The semigroup $S_\eta \in$ **CUHow**.

We were then able to show that S_η is a cancellative semigroup.

Universal property

What do I mean by **canonical**? What is it that makes these semigroups such a natural choice?

Suppose we choose $S \in \mathbf{UHow}_{\mathsf{R}}$. Then for some fixed $n \in \mathbb{N}$ where $n \geq 2$, there exists a homomorphism

$$\phi_\rho : S_\rho \to S.$$

Likewise if we choose any semigroup from

$$\mathbf{UHow}_{\mathsf{L}} \quad \mathbf{CUHow}$$

there exists a corresponding homomorphism from S_λ, S_γ and S_η respectively.

Other semigroup varieties

Equipped with a set of canonical examples of ultra-Howson semigroups, we are yet to construct more examples.

By recalling that $S_\rho = \mathscr{A}^+/\rho^\sharp$, we would like to find all relations ν such that

$$S_\rho/\nu^\sharp \in \mathbf{UHow}_{\mathrm{R}}.$$

As it happens, we already have one such example when we considered the commutative case.

(SC,VG) Conjecture 1

If $\nu = \{([w],[w^2]) : w \in \mathscr{A}^+\}$ then $S_\rho/\nu^\sharp \in \mathbf{UHow}_{\mathrm{R}}$.

Open problems

There are still a number of open problems that we would like to consider.

- Can we get hold of semigroup varieties (or more generally semigroup classes) that **do not** contain an ultra-Howson semigroup?
- Regarding the universal property of the canonical right and left ultra-Howson semigroups, when is the **image** an ultra-Howson semigroup?
- Is there a canonical example of a **non-commutative ultra-Howson** semigroup?

Thanks for listening
Any questions?