The mathematical work of Douglas Munn and its influence

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1. Representations of semigroups
2. Inverse semigroups
3. Semigroup rings

Representations

A representation of a semigroup S (group G) over a field k is a homomorphism $\varphi : S \to \mathrm{End}(V)$ ($\varphi : G \to \mathrm{GL}(V)$) for some vector space of dimension n. n is the degree of φ. Write φ_s for $\varphi(s)$.

φ is null if $\varphi_s = 0$ for all $s \in S$.

Representations $\varphi : S \to \mathrm{End}(V)$, $\psi : S \to \mathrm{End}(W)$ are equivalent if there is an isomorphism $T : V \to W$ such that $\psi_s T = T\varphi_s$ for all $s \in S$. Write $\varphi \sim \psi$.

A subspace W of V invariant under φ if $\varphi_s(w) \in W$ for all $w \in W$ and $s \in S$.

φ is irreducible if it is not null and $\{0\}$ and V are the only subspaces of V invariant under φ.

Representations

Given representations $\varphi : S \to \mathrm{End}(V)$, $\psi : S \to \mathrm{End}(W)$, their direct sum is $\varphi \oplus \psi : S \to V \oplus W$ given by

$$(\varphi \oplus \psi)_s(v, w) = (\varphi_s(v), \psi_s(w))$$

A representation of S is proper if it is not a direct sum with one summand being null.

$\varphi : S \to \mathrm{End}(V)$ is completely reducible if

$$\varphi \sim \varphi^{(1)} \oplus \cdots \oplus \varphi^{(k)}$$

for some irreducible representations $\varphi^{(1)}, \ldots, \varphi^{(k)}$ of S.

Representations

The semigroup algebra $k[S]$ has as its elements finite formal sums $\sum_{s \in S} \alpha_s s$ and multiplication given by

$$(\sum_{s \in S} \alpha_s s)(\sum_{t \in S} \beta_t t) = \sum_{s,t \in S} \alpha_s \beta_t (st).$$

If $k[S]$ is semisimple Artinian, then every proper representation of S is completely reducible .

Theorem

The semigroup algebra $k[S]$ of a finite inverse semigroup S over a field k is semisimple if and only if k has characteristic 0 or a prime not dividing the order of any subgroup of S.

Principal factors

Let $a \in S$:

- J_a denotes the $\mathscr{J}$-class of a;
- $J(a) = S^1 a S^1$;
- $J_a \leqslant J_b$ iff $J(a) \subseteq J(b)$;
- $I(a) = \{b \in J(a) : J_b < J_a\}$.

The principal factors of S are the Rees quotients $J(a)/I(a)$. They are 0-simple, simple or null (all products zero).

Theorem

Let S be a semigroup satisfying the descending chain condition for principal ideals. Suppose also that every principal factor of S is 0-simple or simple. If every representation of every principal factor of S over a field k is completely reducible, then so is every representation of S over k.

Finite semigroups

Let S be a finite semigroup. S has a principal series, i.e., a series

$$S = S_0 \supset S_1 \supset \cdots \supset S_n \supset S_{n+1} = \emptyset$$

where each S_i is an ideal of S and the Rees factors S_i/S_{i+1} are principal factors.
In fact, every principal factor of S is isomorphic to one of the S_i/S_{i+1}.

Theorem

The semigroup algebra $k[S]$ of a finite semigroup S over a field k is semisimple if and only if $k[S_i/S_{i+1}]$ is semisimple for each i.

Characters

Let φ be a representation of a semigroup S. The character of φ is the mapping $\chi : S \to k$ given by $\chi(s) = \operatorname{trace} \varphi_s$ for all $s \in S$.

χ is irreducible if φ is irreducible.

The set $\mathfrak{X}(S)$ of all characters of S forms a ring.

Theorem

Every proper representation over $\mathbb{C}$ of a finite inverse semigroup is completely determined up to equivalence by its character.

Douglas described all irreducible characters of the symmetric inverse monoid $\mathscr{I}_n$ in a 1957 paper.

What happened next?

1. McAlister extends Douglas' results. Notable is:

   Theorem
   Let S be a finite semigroup, $J_1, \ldots, J_n$ the regular $\mathscr{J}$-classes of S and H_i a maximal subgroup of J_i $(i = 1, \ldots, n)$. Then

   $$\mathfrak{X}(S) \cong \mathfrak{X}(H_1) \times \cdots \times \mathfrak{X}(H_n).$$

2. Rhodes/Zalcstein re-work and extend Douglas' work on finite semigroups, and apply to group complexity of finite semigroups.
3. Steinberg uses the inductive groupoid associated with aninverse semigroup.
   Recently, also new approach to representations of general finite semigroups avoiding use of principal factors.
4. Application of representation theory of finite symmetric inverse monoids. (Malandro and Rockmore).

Inverse semigroups: Structure

An inverse semigroup S is:

- bisimple if any two elements are $\mathscr{D}$-related;
- 0-bisimple if it has a zero, and any two non-zero elements are $\mathscr{D}$-related;
- simple if it any two elements are $\mathscr{J}$-related;
- 0-simple if it has a zero, and any two non-zero elements are $\mathscr{J}$-related.

Inverse semigroups: Structure

An ω-chain is a chain $C_\omega = \{e_0, e_1, \ldots, e_n, \ldots\}$ with $e_i \leqslant e_j$ if and only if $j \leqslant i$.

Let A be a monoid, and $\alpha : A \to H_1$ be a homomorphism. Put $\mathrm{BR}(A, \alpha) = \mathbb{N} \times A \times \mathbb{N}$. Define multiplication by

$$(m, a, n)(p, b, q) = (m - n + t, a\alpha^{t-n} b\alpha^{t-p}, q - p + t)$$

where $t = \max\{n, p\}$.

Theorem

Let $S = \mathrm{BR}(A, \alpha)$. Then

1. *S is a simple monoid with identity $(1, 0, 1)$;*
2. *$(m, a, n)\mathscr{D}(p, b, q)$ if and only if $a\mathscr{D}b$;*
3. *(m, a, n) is idempotent if and only if $m = n$ and $a^2 = a$;*
4. *S is inverse if and only if A is inverse;*
5. *If S is inverse, then $E(S) \cong C_\omega \circ E(A)$.*

Inverse semigroups: Structure

Let S be a regular ω-semigroup, that is, an inverse semigroup with $E(S) \cong C_\omega$.

Theorem (Reilly, 1966)

S is bisimple iff $S \cong \mathrm{BR}(G, \alpha)$ for a group G and $\alpha \in \mathrm{End}(G)$.

Theorem (1968)

S is simple iff $S \cong \mathrm{BR}(A, \alpha)$ where A is a finite chain of groups.

This result was also found by Kolchin.
An extension of the theorem gives the structure of a regular ω-semigroup S with minimum ideal $K \neq S$.

Theorem (1968)

The following are equivalent:

1. *S does not have a minimum ideal;*
2. *the idempotents of S are central;*
3. *S is a ω-chain of groups.*

Inverse semigroups: Structure

A semilattice E is:

- uniform (0-uniform) if $Ee \cong Ef$ for all (non-zero) $e, f \in E$;
- subuniform (0-subuniform) if for all (non-zero) $e, f \in E$ there exists $g \in E$ such that $g \leqslant f$ and $Ee \cong Eg$.

Theorem

1. *If S is a bisimple (0-bisimple, simple, 0-simple) inverse semigroup, then $E(S)$ is uniform (0-uniform, subuniform, 0-subuniform).*
2. *If E is a uniform (0-uniform, subuniform, 0-subuniform) semilattice, then there is a bisimple (0-bisimple, simple, 0-simple) inverse semigroup with $E(S) \cong E$.*

Inverse semigroups: Structure

Let S be an inverse semigroup. There is a maximum idempotent-separating congruence μ on S; $\mu \subseteq \mathscr{H}$ and (Howie)

$$a\mu b \text{ if and only if } a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S).$$

S is fundamental if $\mu = \iota$.

The fundamental (or Munn) representation of S is the homomorphism $\alpha : S \to \mathscr{I}_{E(S)}$ given by $a\alpha = \alpha_a$ where $e\alpha_a = a^{-1}ea$.

The Munn semigroup T_E of a semilattice E is the subset of $\mathscr{I}_E$ consisting of all isomorphisms between principal ideals of E.

Inverse semigroups: Structure

Theorem

1. *T_E is an inverse subsemigroup of $\mathscr{I}_E$;*
2. *if $\alpha : S \to \mathscr{I}_{E(S)}$ is the fundamental representation, then $\operatorname{im}\alpha$ is a full subsemigroup of $T_{E(S)}$, $\operatorname{im}\alpha \cong S/\mu$ and $\operatorname{im}\alpha$ is fundamental;*
3. *S is fundamental iff it is isomorphic to a full inverse subsemigroup of $T_{E(S)}$.*

Strategy: describe inverse semigroups by regarding them as extensions of fundamental inverse semigroups. Used by Douglas to describe 0-bisimple inverse semigroups, and applied to get a structure theorem for 0-bisimple (ω, I) inverse semigroups in terms of a maximal subgroup and a semilattice. Reilly's structure theorem is a corollary.
Lallement and Petrich had previously determined the structure of these semigroups using Reilly's theorem.

Inverse semigroups: What happened next

1. McAlister extended the general results for 0-bisimple semigroups of Douglas and Reilly to give a structure theorem for arbitrary 0-bisimple semigroups in terms of groups and 0-uniform semilattices.
2. Hall extends results on the fundamental representation to orthodox semigroups by constructing the *Hall semigroup*, a generalisation of the Munn semigroup.
3. Hall and Nambooripad (independently) extend further to arbitrary regular semigroups.
4. JBF uses Munn semigroup to get fundamental representation of ample semigroups.
5. Gould, Gomes, JBF and El Qallali explore analogues for various classes of weakly ample semigroups and generalisations.

Inverse semigroups: P-semigroups

Let G be group acting by order automorphisms on a partially ordered set X and $Y \subseteq X$. Suppose that

1. Y is an order ideal of X, and a meet semilattice under the induced ordering;
2. $G \centerdot Y = X$;
3. $g \centerdot Y \cap Y \neq \emptyset$ for all $g \in G$.

Put $P = P(G, X, Y) = \{(A, g) \in Y \times G : g^{-1} \cdot A \in Y\}$ and

$$(A, g)(B, h) = (A \wedge g \centerdot B, gh)$$

is an E-unitary inverse semigroup; $Y \cong E(P)$ and $P/\sigma \cong G$. (σ is the minimum group congruence. Inverse S is E-unitary if $e, ea \in E(S) \Rightarrow a \in E(S)$.)

Theorem (McAlister)

1. *Any inverse semigroup is an idempotent-separating homomorphic image of an E-unitary inverse semigroup.*
2. *If S is E-unitary inverse, then $S \cong P(G, X, Y)$ for some G, X, Y.*

Inverse semigroups: P-semigroups

Proving 2.
Given E-unitary S, the crucial question is: what is X?

Douglas: Let $E = E(S)$, $G = S/\sigma$. Define $\preccurlyeq$ on $G \times E$ by:

$(a\sigma, e) \preccurlyeq (b\sigma, f)$ if and only if $\exists c \in R_e \cap Sf$ such that $b\sigma = (a\sigma)(c\sigma)$.

$\preccurlyeq$ is a pre-order. Define ρ on $G \times E$ by:

$(a\sigma, e)\rho(b\sigma, f)$ if and only if $(a\sigma, e) \preccurlyeq (b\sigma, f)$ and $(b\sigma, f) \preccurlyeq (a\sigma, e)$.

ρ is an equivalence on $G \times E$. Put $X = (G \times E)/\rho$ and let $\leqslant$ be the partial order on X induced by $\preccurlyeq$.
The rule: $(a\sigma) \cdot (b\sigma, e) = ((ab)\sigma, e)$ defines an action of G on X by order automorphisms. Put $Y = \{(E, e) : e \in E\} \cong E$. Then Y is an order ideal of X and a lower semilattice under $\leqslant$.

Finally, $S \cong P(G, X, Y)$.

Inverse semigroups: Free inverse semigroups

The free inverse semigroup $\mathrm{FIS}(X)$ on a non-empty set X is an inverse semigroup together with a map $\iota : X \to \mathrm{FIS}(X)$ such that for every inverse semigroup S and every map $\alpha : X \to S$, there is a unique homomorphism $\alpha^* : \mathrm{FIS}(X) \to S$ such that $\iota\alpha^* = \alpha$.

Universal algebra considerations show that free inverse semigroups exist; also the map ι is injective, and $\mathrm{FIS}(X)$ is uniquely determined by X.

The question is: how do we describe its elements? Several answers, but a striking one due to Douglas realises them as certain graphs, now known as Munn trees.

Inverse semigroups: Free inverse semigroups

A ring R is prime if $IJ = 0$ implies $I = 0$ or $J = 0$ where I, J are ideals of R.

A ring R is primitive if it has a faithful simple R-module.

A primitive ring is prime.

Inverse semigroups: Free inverse semigroups

Theorem (Formanek)

Let k be a field. If G is a free group of rank at least 2, then the group algebra $k[G]$ is primitive.

A similar result holds for free monoids/semigroups.

Theorem

For a free inverse semigroup S of finite rank, $k[S]$ is not prime.

Theorem (Pedro Silva)

For a free inverse semigroup S of infinite rank, $k[S]$ is prime.

Theorem (WDM and M.J.Crabb)

For a nontrivial free monoid M and an ideal S of M, tfae:

1. *$k[S]$ is primitive;*
2. *$k[S]$ is prime;*
3. *M has infinite rank.*