Residuated Completely Simple Semigroups

We recall that an ordered semigroup S is residuated if for all $x, y \in S$ there exist

$$x \cdot\!. \, y = max\{z \in S | zy \leq x\} \quad \text{and} \quad x .\!\cdot y = max\{z \in S | yz \leq x\}.$$

We consider a completely simple semigroup $S_x = M(\langle x \rangle; I, \Lambda; P)$, where $\langle x \rangle$ is a cyclic ordered group with $x > 1$ and $p_{11} = x^{-1}$. On this semigroup we define the lexicographic order, the left lexicographic order, the right lexicographic order, the bootlace order, the Cartesian order and the discrete lexicographic order. For each of these orders we determine conditions on matrix P so that the resulting semigroup is ordered. Of these, only the lexicographic and bootlace orders yield residuated semigroups. With the lexicographic order, S_x is orthodox and has a biggest idempotent. With the bootlace order, the maximal idempotents of S_x are identified by specific locations in the sandwich matrix. In the orthodox case there is also a biggest idempotent and, for sandwich matrices of a given size, uniqueness up to ordered semigroup isomorphism is established.