Variants of semigroups - the case study of finite full transformation monoids

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Prime suspects

Mr. Shady Corleone

Violet Moon (special undercover agent)
Now seriously... co-authors

I.D. 

James East
(U. of Western Sydney)
Variants of semigroups

Let \((S, \cdot)\) be a semigroup and \(a \in S\). Given these, one can easily define an alternative product \(\star_a\) on \(S\), namely

\[ x \star_a y = xay. \]

This is the variant \(S^a = (S, \star_a)\) of \(S\) with respect to \(a\).

First mention of variants (as far as we know): Lyapin’s book from 1960 (in Russian).

Magill (1967): Semigroups of functions \(X \to Y\) under an operation defined by

\[ f \cdot g = f \circ \theta \circ g, \]

where \(\theta\) is a fixed function \(Y \to X\). For \(Y = X\), this is exactly a variant of \(T_X\).
History of variants – continued

Hickey (1980s): Variants of general semigroups → a new characterisation of Nambooripad’s order on regular semigroups


G. Y. Tsyaputa (2004/5): variants of finite full transformation semigroups $\mathcal{T}_n$

- classification of non-isomorphic variants
- idempotents, Green’s relations
- analogous questions for $\mathcal{PT}_n$

A more accessible account of her results may be found in the monograph of Ganyushkin & Mazorchuk Classical Finite Transformation Semigroups (Springer, 2009).
Several examples

For a group $G$ and $a \in G$, we always have $G^a \cong G$ via $x \mapsto xa$. The identity element in $G^a$ is $a^{-1}$.

On the other hand, if $S$ the bicyclic monoid, then $a, b \in S$, $a \neq b$ implies $S^a \not\cong S^b$.

If $S$ is a monoid, $a, u, v \in S$, and $u, v$ are units, then $S^{uav} \cong S^a$ via $x \mapsto vxu$.

Thus, for any $a \in T_X$ there exists $e \in E(T_X)$ such that $T^a_X \cong T^e_X$.

A WORD OF CAUTION: If $S$ is a regular semigroup, $S^a$ is not regular in general! However, for regular $S$ and arbitrary $a \in S$, $\text{Reg}(S^a)$ is always a subsemigroup of $S^a$ (Khan & Lawson).
A word of caution, you said…?

Egg-box picture of $T_4^a$ for $a = [1, 2, 3, 3]$
A word of caution, you said…?

Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 1, 3, 3]$

Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 1, 1, 4]$

NBSAN, York, January 14, 2015  Igor Dolinka: Variants of $\mathcal{T}_n$
Three important sets

\[ P_1 = \{ x \in S : \ xa \ R \ x \}, \quad P_2 = \{ x \in S : \ ax \ L \ x \}, \]

\[ P = P_1 \cap P_2 \]

Easy facts:

- \( y \in P_1 \iff L_y \subseteq P_1 \),
- \( y \in P_2 \iff R_y \subseteq P_2 \),
- \( \text{Reg}(S^a) \subseteq P \)
Green’s relations: $R^a, L^a, H^a, D^a$

\[
R^a_x = \begin{cases} 
R_x \cap P_1 & \text{if } x \in P_1 \\
\{x\} & \text{if } x \in S \setminus P_1,
\end{cases}
\]

\[
L^a_x = \begin{cases} 
L_x \cap P_2 & \text{if } x \in P_2 \\
\{x\} & \text{if } x \in S \setminus P_2,
\end{cases}
\]

\[
H^a_x = \begin{cases} 
H_x & \text{if } x \in P \\
\{x\} & \text{if } x \in S \setminus P,
\end{cases}
\]

\[
D^a_x = \begin{cases} 
D_x \cap P & \text{if } x \in P \\
L^a_x & \text{if } x \in P_2 \setminus P_1 \\
R^a_x & \text{if } x \in P_1 \setminus P_2 \\
\{x\} & \text{if } x \in S \setminus (P_1 \cup P_2).
\end{cases}
\]
Group $\mathcal{H}$-classes vs group $\mathcal{H}^a$-classes (in $P$)

Let $S = \mathcal{T}_4$ and $a = [1, 2, 3, 3]$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Is $H_x$ a group $\mathcal{H}$-class of $\mathcal{T}_4$?</th>
<th>Is $H_x$ a group $\mathcal{H}^a$-class of $\mathcal{T}_4^a$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[1, 1, 3, 3]$</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$[4, 2, 2, 4]$</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$[2, 4, 2, 4]$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>$[1, 3, 1, 3]$</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>
...is to conduct a thorough algebraic and combinatorial analysis of $\mathcal{T}_X^a$ where $|X| = n$ and $a$ is a fixed transformation on $X$.

As we noted, we may assume that $a$ is idempotent with $r = \text{rank}(a) < n$,

$$a = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix},$$

so that $a_i \in A_i$ for all $i \in [1, r]$.

Here $A = \text{im}(a) = \{a_1, \ldots, a_r\}$ and $\alpha = \ker(a) = (A_1| \cdots |A_r)$,

with $\lambda_i = |A_i|$. Furthermore, for $I = \{i_1, \ldots, i_m\} \subseteq [1, r]$ we write $\Lambda_I = \lambda_{i_1} \cdots \lambda_{i_m}$ and $\Lambda = \lambda_1 \cdots \lambda_r$. 

Let $B \subseteq X$ and let $\beta$ be an equivalence relation on $X$. We say that $B$ saturates $\beta$ if each $\beta$-class contains at least one element of $B$. Also, we say that $\beta$ separates $B$ if each $\beta$-class contains at most one element of $B$.

\[
P_1 = \{ f \in \mathcal{T}_X : \text{rank}(fa) = \text{rank}(f) \} = \{ f \in \mathcal{T}_X : \alpha \text{ separates im}(f) \}
\]

\[
P_2 = \{ f \in \mathcal{T}_X : \text{rank}(af) = \text{rank}(f) \} = \{ f \in \mathcal{T}_X : A \text{ saturates ker}(f) \}
\]

\[
P = \{ f \in \mathcal{T}_X : \text{rank}(afa) = \text{rank}(f) \} = \text{Reg}(\mathcal{T}_X^a) \leq \mathcal{T}_X^a
\]
Green’s relations in $\mathcal{T}_X^a$ (Tsyaputa, 2004)

\[ R_f^a = \begin{cases} R_f \cap P_1 & \text{if } f \in P_1 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P_1, \end{cases} \]

\[ L_f^a = \begin{cases} L_f \cap P_2 & \text{if } f \in P_2 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P_2, \end{cases} \]

\[ H_f^a = \begin{cases} H_f & \text{if } f \in P \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P, \end{cases} \]

\[ D_f^a = \begin{cases} D_f \cap P & \text{if } f \in P \\ L_f^a & \text{if } f \in P_2 \setminus P_1 \\ R_f^a & \text{if } f \in P_1 \setminus P_2 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus (P_1 \cup P_2). \end{cases} \]
Recall that in $\mathcal{T}_X$, the $D$-classes form a chain:

$$D_n > D_{n-1} > \cdots > D_2 > D_1.$$ 

Each of the $D$-classes $D_{r+1}, \ldots, D_n$ is completely ‘shattered’ into singleton ‘shrapnels’ / $D^a$-classes in $\mathcal{T}_X^a$.

Since all constant maps trivially belong to $P$, $D_1$ is preserved, and remains a right zero band.

For $2 \leq m \leq r$, the class $D_r$ separates into a single regular chunk $D_r \cap P$ and a number of non-regular pieces, as seen on the following picture...
Theorem 4.2 yields an intuitive picture of the Green’s structure of \( T_a \). Recall that the \( D \)-classes of \( T_a \) are precisely the \( D \)-classes contained in such a \( T_a \)-class, namely those of the form \( \{a \} \). Some of these \( D \)-classes in \( T_a \) remain a (regular) \( D \)-class, namely those of the form \( \{a \} \). Now fix some \( r \) such that \( r < n \). The situation is more complicated in \( T_a \). Next, note that (ii) implies (i). Next suppose (iii) holds. Since \( \text{im}(f) \) and \( \text{ker}(f) \) are both ideals in \( T_a \), we may write \( \text{im}(f) = \text{ker}(f) \). The situation is more complicated in \( T_a \). Next, note that (ii) implies (i). Next suppose (iii) holds. Since \( \text{im}(f) \) and \( \text{ker}(f) \) are both ideals in \( T_a \), we may write \( \text{im}(f) = \text{ker}(f) \). The situation is more complicated in \( T_a \). Next, note that (ii) implies (i). Next suppose (iii) holds. Since \( \text{im}(f) \) and \( \text{ker}(f) \) are both ideals in \( T_a \), we may write \( \text{im}(f) = \text{ker}(f) \). The situation is more complicated in \( T_a \).
Order of the $\mathcal{D}^a$-classes

Let $f, g \in \mathcal{T}_X$. Then $D^a_f \leq D^a_g$ in $\mathcal{T}^a_X$ if and only if one of the following holds:

- $f = g$,
- $\text{rank}(f) \leq \text{rank}(aga)$,
- $\text{im}(f) \subseteq \text{im}(ag)$,
- $\text{ker}(f) \supseteq \text{ker}(ga)$.

The maximal $\mathcal{D}^a$-classes are those of the form $D^a_f = \{f\}$ where $\text{rank}(f) > r$. 

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Order of the $\mathcal{D}^a$-classes
The rank of $\mathcal{T}_X^a$

Let $M = \{ f \in \mathcal{T}_X : \text{rank}(f) > r \}$.

Then $\mathcal{T}_X^a = \langle M \rangle$; furthermore, any generating set for $\mathcal{T}_X^a$ contains $M$.

Consequently, $M$ is the unique minimal (with respect to containment or size) generating set of $\mathcal{T}_X^a$, and

$$\text{rank}(\mathcal{T}_X^a) = |M| = \sum_{m=r+1}^{n} S(n, m) \binom{n}{m} m!,$$

where $S(n, m)$ denotes the Stirling number of the second kind.
‘Positioning’ with respect to the regular classes

- If \( f \in P \), then \( D_f^a \leq D_g^a \) if and only if \( \text{rank}(f) \leq \text{rank}(aga) \).
- If \( g \in P \), then \( D_f^a \leq D_g^a \) if and only if \( \text{rank}(f) \leq \text{rank}(g) \).

Consequences:

- The regular \( \mathcal{D}^a \)-classes of \( \mathcal{T}_X^a \) form a chain: \( D_1^a < \cdots < D_r^a \) (where \( D_m^a = \{ f \in P : \text{rank}(f) = m \} \) for \( m \in [1, r] \)).
- ‘Co-ordinatisation’ of the non-regular, ‘fragmented’ \( \mathcal{D}^a \)-classes: if \( \text{rank}(f) = m \leq r \) and \( \text{rank}(afa) = p < m \), then \( D_f^a \) sits below \( D_m^a \) and above \( D_p^a \).
- The ‘crown’: A maximal \( \mathcal{D}^a \)-class \( D_f^a = \{ f \} \) sits above \( D_r^a \) if and only if \( \text{rank}(afa) = r \). The number of such \( \mathcal{D}^a \)-classes is equal to \( (n^{n-r} - r^{n-r})r! \).
Reg($\mathcal{T}_X^a$) – examples

Egg-box diagrams of the regular subsemigroups $P = \text{Reg}(\mathcal{T}_5^a)$ in the cases
(from left to right): $a = [1, 1, 1, 1, 1]$, $a = [1, 2, 2, 2, 2]$, $a = [1, 1, 2, 2, 2]$, $a = [1, 2, 3, 3, 3]$, $a = [1, 2, 2, 3, 3]$, $a = [1, 2, 3, 4, 4]$. 
Do you see what I am seeing???

Egg-box diagrams of $\mathcal{T}_3$ (left) and $\text{Reg}(\mathcal{T}_5^a)$ for $a = [1, 2, 2, 3, 3]$ (right).
No, this is not just a coincidence...!

\[ \mathcal{T}(X, A) = \{ f \in \mathcal{T}_X : \text{im}(f) \subseteq A \} \]

\[ \mathcal{T}(X, \alpha) = \{ f \in \mathcal{T}_X : \text{ker}(f) \supseteq \alpha \} \]

– transformation semigroups with restricted range (Sanwong & Sommanee, 2008), and restricted kernel (Mendes-Gonçalves & Sullivan, 2010).

Fact:

\[ \text{Reg}(\mathcal{T}(X, A)) = \mathcal{T}(X, A) \cap P_2 \]

\[ \text{Reg}(\mathcal{T}(X, \alpha)) = \mathcal{T}(X, \alpha) \cap P_1 \]
Structure Theorem – Part 1

\[ \psi : f \mapsto (fa, af) \]

is a well-defined embedding of \( \text{Reg}(\mathcal{T}_X^a) \) into the direct product \( \text{Reg}(\mathcal{T}(X, A)) \times \text{Reg}(\mathcal{T}(X, \alpha)) \). Its image consists of all pairs \((g, h)\) such that

\[ \text{rank}(g) = \text{rank}(h) \quad \text{and} \quad g|_A = (ha)|_A. \]

Thus \( \text{Reg}(\mathcal{T}_X^a) \) is a subdirect product of \( \text{Reg}(\mathcal{T}_X^a) \) and \( \text{Reg}(\mathcal{T}(X, \alpha)) \).
The maps

\[ \phi_1 : \text{Reg}(\mathcal{T}(X, A)) \rightarrow \mathcal{T}_A : g \mapsto g|_A \]

\[ \phi_2 : \text{Reg}(\mathcal{T}(X, \alpha)) \rightarrow \mathcal{T}_A : g \mapsto (ga)|_A \]

are epimorphisms, and the following diagram commutes:

Further, the induced map \( \phi = \psi_1 \phi_1 = \psi_2 \phi_2 = \text{Reg}(\mathcal{T}^a_X) \rightarrow \mathcal{T}_A \) is an epimorphism that is ‘group / non-group preserving’.
Size and rank of $P = \text{Reg}(\mathcal{T}_X^a)$

$$|P| = \sum_{m=1}^{r} m! m^{n-r} S(r, m) \sum_{I \in \left[1, r\right]} \Lambda_I.$$ 

Let $D$ be the top (rank-$r$) $\mathcal{D}^a$-class of $P$.

$$\text{rank}(P) = \text{rank}(D) + \text{rank}(P : D) = r^{n-r} + 1$$
The idempotent generated subsemigroup $\left\langle E_a(T^a_X) \right\rangle_a$

- $E_a(T^a_X) = \{ f \in T_X : (af)|_{\text{im}(f)} = \text{id}|_{\text{im}(f)} \}.$

- $|E_a(T^a_X)| = \sum_{m=1}^{r} m^{n-m} \sum_{I \in \binom{[1,r]}{m}} \Lambda_I.$

- We obtain a pleasing generalisation of celebrated Howie’s Theorem:

$$\mathcal{E}^a_X = \left\langle E_a(T^a_X) \right\rangle_a = E_a(D) \cup (P \setminus D).$$
The idempotent generated subsemigroup \( \langle E_a(T_X^a) \rangle_a \)

\[ \text{rank}(E_X^a) = \text{idrank}(E_X^a) = r^{n-r} + \rho_r, \]
where \( \rho_2 = 2 \) and \( \rho_r = \binom{r}{2} \) if \( r \geq 3 \).

The number of idempotent generating sets of \( E_X^a \) of the minimal possible size is

\[
\left[ (r - 1)^{n-r} \Lambda \right]^{\rho_r} \Lambda! S(r^{n-r}, \Lambda) \sum_{\Gamma \in \mathbb{T}_r} \frac{1}{\lambda_1^{d_{\Gamma}^+(1)} \cdots \lambda_r^{d_{\Gamma}^+(r)}}.
\]

where \( \mathbb{T}_r \) is the set of all strongly connected tournaments on \( r \) vertices.
The ideals of $P$

- The ideals of $P$ are precisely

$$l^a_m = \{ f \in P : \text{rank}(f) \leq m \}$$

for $m \in [1, r]$.

- They are all idempotent generated (by $E_a(D^a_m)$) except $P = l^a_r$ itself.

- 

$$\text{rank}(l^a_m) = \text{idrank}(l^a_m) = \begin{cases} m^{n-r}S(r, m) & \text{if } 1 < m < r \\ n & \text{if } m = 1. \end{cases}$$
Future work

- Conduct an analogous study for variants of:
  - full linear (matrix) monoids
  - symmetric inverse semigroups
  - various diagram semigroups (partition, (partial) Brauer, (partial) Jones, wire, Kaufmann, ...)
  - ...

- Consider an ‘Ehresmann-style’ defined small (semi)category (aka partial monoid / semigroup) $S$. One can turn each hom-set $S_{ij}$ ($i$ - domain, $j$ - codomain) into a semigroup by fixing a ‘sandwich’ element $a \in S_{ji}$ and defining

  $$x \star y = x \circ a \circ y.$$  

These sandwich semigroups generalise the variants.
  - applicable to functions, matrices, diagrams, ...
THANK YOU!

Questions and comments to:
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Further information may be found at:
http://people.dmi.uns.ac.rs/~dockie