Idempotent generators in finite partition monoids

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with Robert Gray (University of East Anglia)

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Joint work with Bob Gray

James East  
Idempotent generators in finite partition monoids
0. Outline

1. Transformation semigroups
   - Singular part
   - Ideals

2. Partition monoids

3. Brauer monoids

4. Jones monoids

5? Regular \(*\)-semigroups
Don’t mention the cri%$et

James East  Idempotent generators in finite partition monoids
1. Transformation Semigroups

Let

- $n$ be a positive integer
- $n = \{1, \ldots, n\}$
- $S_n = \{\text{permutations } n \to n\}$ — symmetric group
- $T_n = \{\text{functions } n \to n\}$ — transformation semigroup
- $T_n \setminus S_n = \{\text{non-invertible functions } n \to n\}$ — singular ideal
1. Transformation Semigroups

**Theorem (Howie, 1966)**

- \( \mathcal{T}_n \setminus S_n \) is idempotent generated.
- \( \mathcal{T}_n \setminus S_n = \langle e_{ij}, e_{ji} : 1 \leq i < j \leq n \rangle \).

![Diagram of \( e_{ij} \) and \( e_{ji} \)]

**Theorem (Howie, 1978)**

- \( \text{rank}(\mathcal{T}_n \setminus S_n) = \text{idrank}(\mathcal{T}_n \setminus S_n) = \binom{n}{2} = \frac{n(n-1)}{2} \).
1. Transformation Semigroups

Theorem (Howie, 1978)

For \( X \subseteq \{ e_{ij}, e_{ji} : 1 \leq i < j \leq n \} \), define a di-graph \( \Gamma_X \) by

- \( V(\Gamma_X) = \mathbb{N} \), and
- \( E(\Gamma_X) = \{(i, j) : e_{ij} \in X\} \).

Then \( T_n \cap S_n = \langle X \rangle \) iff \( \Gamma_X \) is strongly connected and complete.

\[ T_3 \cap S_3 = \langle e_{12}, e_{23}, e_{31} \rangle \]

\[ T_3 \cap S_3 \neq \langle e_{12}, e_{23}, e_{13} \rangle \]
Theorem (Howie, 1978 and Wright, 1970)

The minimal idempotent generating sets of $\mathcal{T}_n \setminus S_n$ are in one-one correspondence with the strongly connected labelled tournaments on $n$ nodes.

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The ideals of $\mathcal{T}_n$ are $I_r = \{ \alpha \in \mathcal{T}_n : |\text{im}(\alpha)| \leq r \}$ for $1 \leq r \leq n$.

**Theorem (Howie and McFadden, 1990)**

If $2 \leq r \leq n - 1$, then $I_r$ is idempotent generated, and

$$\text{rank}(I_r) = \text{idrank}(I_r) = S(n, r),$$

a Stirling number of the second kind.

- $I_{n-1} = \mathcal{T}_n \setminus S_n$ and $S(n, n - 1) = \binom{n}{2}$.
- $\text{rank}(I_1) = \text{idrank}(I_1) = |I_1| = n$ — right zero semigroup.
- Similar results for matrix semigroups (and others).
- Today: diagram monoids.
2. Partition Monoids

- Let \( n = \{1, \ldots, n\} \) and \( n' = \{1', \ldots, n'\} \).

- The *partition monoid* on \( n \) is
  \[ \mathcal{P}_n = \{\text{set partitions of } n \cup n'\} \]
  \[ \equiv \{\text{(equiv. classes of) graphs on vertex set } n \cup n'\}\}. \]

- Eg: \( \alpha = \left\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 6'\}, \{2', 3'\}, \{5'\}\right\} \in \mathcal{P}_6 \)
Let $\alpha, \beta \in \mathcal{P}_n$. To calculate $\alpha \beta$:

1. connect bottom of $\alpha$ to top of $\beta$,  
2. remove middle vertices and floating components,  
3. smooth out resulting graph to obtain $\alpha \beta$.

![Diagram](image)

The operation is associative, so $\mathcal{P}_n$ is a semigroup (monoid, etc).

- What can we say about idempotents and ideals of $\mathcal{P}_n$?
2. Partition Monoids — Submonoids of $\mathcal{P}_n$

- $B_n = \{ \alpha \in \mathcal{P}_n : \text{blocks of } \alpha \text{ have size 2} \}$ — Brauer monoid
  
  \[
  \begin{array}{cc}
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \end{array}
  \]
  
  $\in B_5$

- $S_n = \{ \alpha \in B_n : \text{blocks of } \alpha \text{ hit } n \text{ and } n' \}$ — symmetric group
  
  \[
  \begin{array}{cc}
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \end{array}
  \]
  
  $\in S_5$

- $J_n = \{ \alpha \in B_n : \alpha \text{ is planar} \}$ — Jones monoid
  
  \[
  \begin{array}{cc}
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \circ & \circ \\
  \end{array}
  \]
  
  $\in J_5$

What can we say about idempotents and ideals of $\mathcal{P}_n$? $B_n$? $J_n$?
2. Partition Monoids

**Theorem (E, 2011)**

- $\mathcal{P}_n \setminus \mathcal{S}_n$ is idempotent generated.
- $\mathcal{P}_n \setminus \mathcal{S}_n = \langle t_r, t_{ij} : 1 \leq r \leq n, 1 \leq i < j \leq n \rangle$.

$$t_r = \begin{array}{ccccccc}
1 & \cdots & r & \cdots & n \\
\cdot & & \cdot & & \cdot
\end{array} \quad t_{ij} = \begin{array}{ccccccc}
1 & \cdots & i & \cdots & j & \cdots & n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

- $\text{rank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \binom{n+1}{2} = \frac{n(n+1)}{2}$. 

Any minimal idempotent generating set for $P_n \setminus S_n$ is a subset of

$$\{t_r : 1 \leq r \leq n\} \cup \{t_{ij}, e_{ij}, e_{ji}, f_{ij}, f_{ji} : 1 \leq i < j \leq n\}.$$
2. Partition Monoids

Let $\Gamma_n$ be the di-graph with vertex set

$$V(\Gamma_n) = \{A \subseteq n : |A| = 1 \text{ or } |A| = 2\}$$

and edge set

$$E(\Gamma_n) = \{(A, B) : A \subseteq B \text{ or } B \subseteq A\}.$$

$\Gamma_5$ (with loops omitted)
2. Partition Monoids

A subgraph $H$ of a di-graph $G$ is a permutation subgraph if $V(H) = V(G)$ and the edges of $H$ induce a permutation of $V(G)$.

A permutation subgraph of $\Gamma_n$ is determined by:

- a permutation of a subset $A$ of $n$ with no fixed points or 2-cycles ($A = \{2, 3, 5\}$, $2 \mapsto 3 \mapsto 5 \mapsto 2$), and
- a function $n \setminus A \to n$ with no 2-cycles ($1 \mapsto 4$, $4 \mapsto 4$).
2. Partition Monoids

Theorem (E+Gray, 2013)

The minimal idempotent generating sets of $\mathcal{P}_n \setminus S_n$ are in one-one correspondence with the permutation subgraphs of $\Gamma_n$.

The number of minimal idempotent generating sets of $\mathcal{P}_n \setminus S_n$ is equal to

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n,n-k},$$

where $a_0 = 1$, $a_1 = a_2 = 0$, $a_{k+1} = ka_k + k(k-1)a_{k-2}$, and

$$b_{n,k} = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^i \binom{k}{2i} (2i - 1)!!n^{k-2i}.$$

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The ideals of $\mathcal{P}_n$ are

$$I_r = \{ \alpha \in \mathcal{P}_n : \alpha \text{ has } \leq r \text{ transverse blocks} \}$$

for $0 \leq r \leq n$.

**Theorem (E+G, 2013)**

If $0 \leq r \leq n - 1$, then $I_r$ is idempotent generated, and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \sum_{j=r}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) S(j, r) B_{n-j} = \sum_{j=r}^{n} S(n, j) \left( \begin{array}{c} j \\ r \end{array} \right),$$

where $B_k$ is the $k$th Bell number.
Let $\Lambda_n$ be the di-graph with vertex set

$$V(\Lambda_n) = \{A \subseteq \mathbb{n} : |A| = 2\}$$

and edge set

$$E(\Lambda_n) = \{(A, B) : A \cap B \neq \emptyset\}.$$
3. Brauer Monoids

Theorem (E+G, 2013)

The minimal idempotent generating sets of $\mathcal{B}_n \setminus \mathcal{S}_n$ are in one-one correspondence with the permutation subgraphs of $\Lambda_n$.

No formula is known for the number of minimal idempotent generating sets of $\mathcal{B}_n \setminus \mathcal{S}_n$ (yet). Very hard!

$$
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\hline
 1 & 1 & 1 & 6 & 265 & 126,140 & 855,966,441 & \text{????} & \cdots
\end{array}
$$

There are (way) more than $(n - 1)! \cdot (n - 2)! \cdot \cdots \cdot 3! \cdot 2! \cdot 1!$.

- Thanks to James Mitchell for $n = 5, 6$.
- Partition monoids are now on GAP!
- Semigroups package: tinyurl.com/semigroups
The ideals of $B_n$ are

$$I_r = \{ \alpha \in B_n : \alpha \text{ has } \leq r \text{ transverse blocks} \}$$

for $0 \leq r = n - 2k \leq n$.

**Theorem (E+G, 2013)**

If $0 \leq r = n - 2k \leq n - 2$, then $I_r$ is idempotent generated and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \binom{n}{2k}(2k - 1)!! = \frac{n!}{2^kk!r!}.$$
Let $\Xi_n$ be the di-graph with vertex set

$$V(\Xi_n) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$$

and edge set

$$E(\Xi_n) = \{(A, B) : A \cap B \neq \emptyset\}.$$
4. Jones Monoids

Theorem (E+G, 2013)

The minimal idempotent generating sets of $J_n \setminus \{1\}$ are in one-one correspondence with the permutation subgraphs of $\Xi_n$.

The number of minimal idempotent generating sets of $J_n \setminus \{1\}$ is $F_n$, the $n$th Fibonacci number.

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The ideals of $\mathcal{J}_n$ are

$$l_r = \{ \alpha \in \mathcal{J}_n : \alpha \text{ has } \leq r \text{ transverse blocks} \}$$

for $0 \leq r = n - 2k \leq n$.

**Theorem (E+G, 2013)**

If $0 \leq r = n - 2k \leq n - 2$, then $l_r$ is idempotent generated and

$$\text{rank}(l_r) = \text{idrank}(l_r) = \frac{r + 1}{n + 1} \binom{n + 1}{k}.$$
Values of $\text{rank}(I_r) = \text{idrank}(I_r)$:

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Definition

\((S, \cdot, ^*)\) is a regular \(*\)-semigroup if \((S, \cdot)\) is a semigroup and

\[
\begin{align*}
    s^{**} &= s, & (st)^* &= t^*s^*, & ss^*s &= s \quad \text{(and } s^*ss^* = s^*). 
\end{align*}
\]

Examples

- groups and inverse semigroups, where \(s^* = s^{-1}\)
- \(P_n\), where \(\alpha^* = \alpha\) turned upside down
- \(B_n, J_n, S_n\)
- Not \(T_n\) — \(J\)-classes must be square
5. Regular \(*\)-semigroups

*Green’s relations* on a semigroup $S$ are defined, for $x, y \in S$, by

- $x \mathbin{\mathcal{L}} y$ if and only if $S^1 x = S^1 y$,
- $x \mathbin{\mathcal{R}} y$ if and only if $x S^1 = y S^1$,
- $x \mathbin{\mathcal{J}} y$ if and only if $S^1 x S^1 = S^1 y S^1$,
- $x \mathbin{\mathcal{H}} y$ if and only if $x \mathbin{\mathcal{L}} y$ and $x \mathbin{\mathcal{R}} y$.

Within a $\mathcal{J}$-class $J(x)$ in a finite semigroup:

- the $\mathcal{R}$-class $R(x)$
- the $\mathcal{L}$-class $L(x)$
- the $\mathcal{H}$-class $H(x)$
The $\mathcal{J}$-classes of a semigroup $S$ are partially ordered:

- $J(x) \leq J(y)$ iff $x \in S^1 y S^1$. 

Diagram: 

- A complex diagram illustrating the partial ordering of $\mathcal{J}$-classes with nodes and directed edges.
5. Regular $\ast$-semigroups

The $\mathcal{J}$-classes of a semigroup $S$ are partially ordered:

- $J(x) \leq J(y)$ iff $x \in S^1yS^1$.

If $S$ is $\mathcal{P}_n \setminus S_n$ or $\mathcal{B}_n \setminus S_n$ or $\mathcal{I}_n \setminus \{1\}$, then:

- $S$ is a regular $\ast$-semigroup,
- $S$ is idempotent generated,
- the $\mathcal{J}$-classes form a chain $J_1 < \cdots < J_k$,
- $J_r \subseteq \langle J_{r+1} \rangle$ for each $r$. 

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Idempotent generators in finite partition monoids
5. Regular $\ast$-semigroups — $\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6$ (thanks to GAP)
Theorem (applies to $\mathcal{P}_n \setminus S_n$ and $\mathcal{B}_n \setminus S_n$ and $\mathcal{J}_n \setminus \{1\}$)

Let $S$ be a finite regular $\ast$-semigroup and suppose

- $S$ is idempotent generated,
- the $\mathcal{J}$-classes of $S$ form a chain $J_1 < \cdots < J_k$,
- $J_r \subseteq \langle J_{r+1} \rangle$ for each $r$.

Then

- the ideals of $S$ are the sets $I_r = \langle J_r \rangle = J_1 \cup \cdots \cup J_r$,
- the ideals of $S$ are idempotent generated,
- $\text{rank}(I_r) = \text{idrank}(I_r) = \text{the number of } \mathcal{R}\text{-classes in } J_r$. 
If $J$ is a $\mathcal{J}$-class of a semigroup $S$, we may form the *principle factor*

$$J^\circ = J \cup \{0\}$$

with product $s \circ t = \begin{cases} st & \text{if } s, t, st \in J \\ 0 & \text{otherwise.} \end{cases}$

**Lemma (applies to $\mathcal{P}_n \setminus S_n$ and $\mathcal{B}_n \setminus S_n$ and $\mathcal{J}_n \setminus \{1\}$)**

If $S = \langle J \rangle$ where $J$ is a $\mathcal{J}$-class, then

$$\text{rank}(S) = \text{rank}(J^\circ).$$

Further, $S$ is idempotent generated iff $J^\circ$ is, and

$$\text{idrank}(S) = \text{idrank}(J^\circ).$$

Any minimal (idempotent) generating set for $S$ is contained in $J$. 

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Idempotent generators in finite partition monoids
Proposition

Let

- $S$ be a regular $\ast$-semigroup,
- $E(S) = \{s \in S : s^2 = s\}$ — idempotents of $S$,
- $P(S) = \{s \in S : s^2 = s = s^*\}$ — projections of $S$.

Then

- $E(S) = P(S)^2$,
- $\langle E(S) \rangle = \langle P(S) \rangle$,
- $S$ is idempotent generated iff it is projection generated,
- each $\mathcal{R}$-class (and $\mathcal{L}$-class) contains exactly one projection.
Consider the projections of some finite regular $\ast$-semigroup $J^\circ$:

We create a graph $\Gamma(J^\circ)$. 

\[ 0 = pr = rp = qr = rq = qs = sq \]
5. Regular $\ast$-semigroups — minimal generating sets

**Definition**

The graph $\Gamma(J^\circ)$ has:
- vertices $P(J) = \{\text{non-zero projections}\}$,
- edges $p \to q$ iff $pq \in J$.

If $S = \langle J \rangle$ is a finite idempotent generated regular $\ast$-semigroup, we define $\Gamma(S) = \Gamma(J^\circ)$.

**Theorem**

A subset $F \subseteq E(J)$ determines a subgraph $\Gamma_F(S)$ with

$$V(\Gamma_F(S)) = P(J) \quad \text{and} \quad E(\Gamma_F(S)) = \{ p \to q : pq \in F \}.$$

The set $F$ is a minimal (idempotent) generating set for $S$ iff $\Gamma_F(S)$ is a permutation subgraph.
Thanks for listening

Thank You!!