Semigroups with skeletons and Zappa-Szép products

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Definitions and basics

Restriction semigroups with skeletons

Special $\tilde{D}_E$-simple restriction monoids and Zappa-Szép products

Deduction and applications to bisimple inverse monoids
Definitions and basics

The relations $\tilde{R}_E$ and $\tilde{L}_E$

Let $S$ be a semigroup and $E$ be a distinguished set of idempotents. The relation $\tilde{R}_E$ is defined by $a \tilde{R}_E b$ if and only if for all $e \in E$,

$$ea = a \iff eb = b.$$ 

The relation $\tilde{L}_E$ is dual.

Note that:

- The relations $\tilde{R}_E$ and $\tilde{L}_E$ are equivalence relations.

- $R \subseteq \tilde{R}_E$ and $L \subseteq \tilde{L}_E$.

The relation $\tilde{H}_E$ is the intersection of $\tilde{R}_E$ and $\tilde{L}_E$ and the relation $\tilde{D}_E$ is the join of $\tilde{R}_E$ and $\tilde{L}_E$. 

A semigroup $S$ satisfies the congruence condition $(C)$ if $\tilde{R}_E$ is a left congruence and $\tilde{L}_E$ is a right congruence.

We will denote the $\tilde{R}_E$-class ($\tilde{L}_E$-class, $\tilde{H}_E$-class) of any $a \in S$ by $\tilde{R}_E^a$ ($\tilde{L}_E^a$, $\tilde{H}_E^a$).

If $S$ satisfies $(C)$, then $\tilde{H}_E^e$ is a monoid with identity $e$, for any $e \in E$.

**Weakly $E$-abundant semigroups**

A semigroup $S$ with $E \subseteq E(S)$ is said to be *weakly $E$-abundant* if every $\tilde{R}_E$- and every $\tilde{L}_E$-class of $S$ contains an idempotent of $E$.

**$E$-regular elements**

Let $S$ be a semigroup and $E \subseteq E(S)$. We say that an element $c \in S$ is *$E$-regular* if $c$ has an inverse $c^\circ$ such that $cc^\circ, c^\circ c \in E$. 
**Lemma** Let $S$ be a semigroup with $(C)$ and suppose $S$ has an $E$-regular element $c$ such that

$$cc^o = e, \ c^o c = f$$

Then the right translations

$$\rho_c : \tilde{L}_E^e \to \tilde{L}_E^f \quad \text{and} \quad \rho_c^o : \tilde{L}_E^f \to \tilde{L}_E^e$$

are mutually inverse $\tilde{R}_E$-class preserving bijections and the left translations

$$\lambda_c^o : \tilde{R}_E^e \to \tilde{R}_E^f \quad \text{and} \quad \lambda_c : \tilde{R}_E^f \to \tilde{R}_E^e$$

are mutually inverse $\tilde{L}_E$-class preserving bijections.
The following “egg” box picture helps us to understand the above Lemma.

\[ \rho_c \]
\[ \lambda_{c^0} \]

\[ \begin{array}{c|c}
  e & c \\
  \hline
  u & uc \\
  \hline
  c^o & f \\
\end{array} \]

**Corollary** Let \( S \) be a semigroup with \( (C) \). Let \( c \) be an \( E \)-regular element of \( S \) such that

\[ cc^o = e, \ c^o c = f. \]

Then \( \tilde{H}_E^e \cong \tilde{H}_E^f \).
Let $S$ be a semigroup and $E \subseteq E(S)$. Suppose every $\tilde{\mathcal{H}}_E$-class contains an $E$-regular element. Then

1. $S$ is weakly $E$-abundant;

2. if $S$ has (C), then $\tilde{\mathcal{R}}_E \circ \tilde{\mathcal{L}}_E = \tilde{\mathcal{L}}_E \circ \tilde{\mathcal{R}}_E$ (so that $\tilde{\mathcal{D}}_E = \tilde{\mathcal{R}}_E \circ \tilde{\mathcal{L}}_E$);

3. if $a, b \in S$ with $a \tilde{\mathcal{D}}_E b$, then $|\tilde{\mathcal{H}}^a_E| = |\tilde{\mathcal{H}}^b_E|$;

4. if $E$ is a band and $\tilde{\mathcal{H}}_E$ is a congruence, then for $k \in S$ and $k \tilde{\mathcal{H}}_E k^2$, $E \cap \tilde{\mathcal{H}}^k_E \neq \emptyset$. 

Rida-E Zenab

Semigroups with skeletons and Zappa-Szép products
A subsemigroup $M$ of a semigroup $S$ has the right congruence extension property if for any right congruence $\rho$ on $M$ we have

$$\rho = \bar{\rho} \cap (M \times M)$$

where $\bar{\rho} = \langle \rho \rangle$ is right congruence on $S$.

**Lemma** Let $S$ be a weakly $E$-abundant semigroup with $(C)$. Suppose that $\tilde{\mathcal{H}}_E$ is a congruence. Let $e \in E$. Then $M = H^e_E$ has the right congruence extension property.
We say that a congruence $\rho$ on $M$ is closed under conjugation if for $u, v \in M$ with $u \rho v$ and for any $c \in S$, with $cc^\circ, c^\circ c \in E$ and $cuc^\circ, cvc^\circ \in M$,

$$cuc^\circ \rho cvc^\circ$$

**Lemma** Let $S$ be a semigroup with (C) such that every $\tilde{\mathcal{H}}_E$-class contains an $E$-regular element, $E$ is a band and $\tilde{\mathcal{H}}_E$ is a congruence. Let $e \in E$ and $M = \tilde{\mathcal{H}}_E^e$. Let $\rho$ be a congruence on $M$. Then

$$\rho = \bar{\rho} \cap (M \times M)$$

if and only if $\rho$ is closed under conjugation.
Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by $^+$. The identities that define a left restriction semigroup $S$ are:

$$a^+ a = a, a^+ b^+ = b^+ a^+, (a^+ b)^+ = a^+ b^+, ab^+ = (ab)^+ a.$$ 

We put

$$E = \{ a^+ : a \in S \},$$

then $E$ is a semilattice known as the semilattice of projections of $S$.

Dually right restriction semigroups form a variety of unary semigroups. In this case the unary operation is denoted by $^\ast$.

A semigroup is restriction, if it is both left and right restriction with same semilattice of projections.
If a restriction semigroup $S$ has an identity element $1$, then
\[ 1^+ = 1^* = 1. \]

Such a restriction semigroup is called a restriction monoid.

We consider special classes of restriction semigroups that consists of single $\mathcal{D}_E$-classes. Such semigroups are called $\mathcal{D}_E$-simple semigroups.
We say that a subset $V$ of $W$, where $W \subseteq S$ and $W$ is a union of $\tilde{H}_E$-classes, is an $\tilde{H}_E$-transversal of $W$ if

$$|V \cap \tilde{H}_E^a| = 1 \quad \text{for all } a \in W.$$ 

**Example 1**

Let $S = BR(M, \theta)$, where $M$ is a monoid. Then $(0, 1, 0)$ is the identity of $S$ and

$$\tilde{L}_E^{(0,1,0)} = \{(a, l, 0) : a \in \mathbb{N}^0, l \in M\},$$

$$\tilde{R}_E^{(0,1,0)} = \{(0, m, a) : a \in \mathbb{N}^0, m \in M\}$$

are $\tilde{L}_E$- and $\tilde{R}_E$-classes of the identity respectively. Let

$$L = \{(a, 1, 0) : a \in \mathbb{N}^0\}.$$ 

Clearly $L$ is a submonoid $\tilde{H}_E$-transversal of $\tilde{L}_E^{(0,1,0)}$. 

Rida-E Zenab

Semigroups with skeletons and Zappa-Szép products
Definition

Let $S$ be a semigroup with $E \subseteq E(S)$. Let $U$ be a subset of $S$ consisting of $E$-regular elements, where $E \subseteq U$. If $U$ intersects every $\tilde{H}_E$-class of $S$ ($U$ is an $\tilde{H}_E$-transversal of $S$), then $U$ is a (combinatorial) inverse skeleton of $S$. If in addition $U$ is a subsemigroup, then $U$ is a (combinatorial) inverse $S$-skeleton.

Example

Let $S = B^\circ(M, I)$ be a Brandt semigroup, where $M$ is a monoid. Then

$$U = \{(i, 1, j) : i \in I\} \cup \{0\}$$

is a combinatorial inverse $S$-skeleton of $S$. 
Theorem 1 Let $S$ be a $\tilde{D}_E$-simple restriction monoid with $\tilde{R}_E \circ \tilde{L}_E = \tilde{L}_E \circ \tilde{R}_E$. Suppose there is a submonoid $\tilde{H}_E$-transversal $L$ of $\tilde{L}_E$ such that every $c \in L$ is $E$-regular and for all $c \in L$, $e \in E$ we have $cec^\circ, c^\circ ec \in E$. Let

$$R = \{c^\circ : c \in L\}.$$ 

Then $R$ is a submonoid $\tilde{H}_E$-transversal of $\tilde{R}_E^1$.  

Suppose in addition that $RL \subseteq R \cup L$. Then $U = \langle R \cup L \rangle = LR$ and $U$ is a combinatorial inverse $S$-skeleton for $S$. 

Example

Going back to Example 1 let \((a, 1, 0) \in L\). Putting

\[(a, 1, 0)^\circ = (0, 1, a)\]

we have that \((a, 1, 0)^\circ\) is an inverse of \((0, 1, a)\). Set

\[R = \{(a, 1, 0)^\circ : (a, 1, 0) \in L\}\]

We note that \(R\) is a submonoid \(\tilde{\mathcal{H}}_E\) transversal of \(\tilde{R}_E^{(0,1,0)}\). Also \(RL \subseteq R \cup L\).
Then

\[ U = \{(a, 1, b) : a, b \in \mathbb{N}^0\} \]

is a combinatorial inverse \( S \)-skeleton of \( S \).

Example

Let \( S = BR(M, \mathbb{Z}, \theta) \) be extended Bruck-Reilly extension of monoid \( M \). The semigroup operation on \( S \) is defined by the rule:

\[
(k, s, l)(m, t, n) = \begin{cases} 
(k - l + m, (s)\theta^{m-l}t), n), & \text{if } l < m; \\
(k, st, n), & \text{if } l = m; \\
(k, s(t)\theta^{l-m}, n - m + l), & \text{if } l > m.
\end{cases}
\]

for \( k, l, m, n \in \mathbb{Z} \) and \( s, t \in M \). Then \( S \) has an inverse skeleton
Example

Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$ be a strong semilattice $Y$ of monoids $S_\alpha$, where

$$\chi_{\alpha,\beta} : S_\alpha \rightarrow S_\beta$$

is a monoid homomorphism such that

1. $\chi_{\alpha,\alpha} = 1_{S_\alpha}$,
2. $\chi_{\alpha,\beta} \chi_{\beta,\gamma} = \chi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$

On $S = \bigcup_{\alpha \in Y} S_\alpha$, multiplication is defined by

$$ab = (a \chi_{\alpha,\alpha \beta})(b \chi_{\beta,\alpha \beta}) \quad a \in S_\alpha, b \in S_\beta.$$  

Let $e_\alpha$ be the identity of $S_\alpha$. Then $E = \{e_\alpha : \alpha \in Y\}$ is a semilattice, $S$ is a restriction semigroup with respect to $E$ and the $\tilde{\mathcal{H}}_E$-classes are the $S_\alpha$’s. Then $E$ is an inverse $S$-skeleton.
Definition

Let $S$ be a $\tilde{\mathcal{D}}_E$-simple restriction monoid. We say that $S$ is special if $\tilde{R}_E \circ \tilde{L}_E = \tilde{L}_E \circ \tilde{R}_E$ and there is a submonoid $\tilde{\mathcal{H}}_E$-transversal $L$ of $\tilde{L}_E^1$ such that every $c \in L$ is $E$-regular and for all $c \in L$, $e \in E$ we have $cec^\circ, c^\circ ec \in E$.

If $S$ is a special $\tilde{\mathcal{D}}_E$-simple restriction monoid, then by \( R = \{ c^\circ : c \in L \} \) is a submonoid $\tilde{\mathcal{H}}_E$-transversal of $\tilde{R}_E^1$. 
Let $S$ and $T$ be semigroups and suppose that we have maps
\[ T \times S \to S, \quad (t, s) \mapsto t \cdot s \]
\[ T \times S \to T, \quad (t, s) \mapsto t^s \]
such that for all $s, s' \in S, t, t' \in T$, the following hold:

1. **ZS1** $tt' \cdot s = t \cdot (t' \cdot s)$;
2. **ZS2** $t \cdot (ss') = (t \cdot s)(t^s \cdot s')$;
3. **ZS3** $(t^s)^{s'} = t^{ss'}$;
4. **ZS4** $(tt')^s = t^{t' \cdot s} t'^s$.

Define a binary operation on $S \times T$ by
\[ (s, t)(s', t') = (s(t \cdot s'), t^{s'} t'). \]
Then $S \times T$ is a semigroup, known as the Zappa-Szép product of $S$ and $T$ and denoted by $S \Join T$.

If $S$ and $T$ are monoids then we insist that the following four axioms also hold:

\begin{align*}
\text{ZS5} \quad t \cdot 1_S &= 1_S; \\
\text{ZS6} \quad t^{1_S} &= t; \\
\text{ZS7} \quad 1_T \cdot s &= s; \\
\text{ZS8} \quad 1_T^s &= 1_T.
\end{align*}

Then $S \Join T$ is monoid with identity $(1_S, 1_T)$. 
Kunze discovered that the Bruck-Reilly extension of a monoid \( BR(S, \theta) \) is the Zappa-Szép product of \( \mathbb{N}^0 \) under addition and the semidirect product \( \mathbb{N}^0 \ltimes S \), where multiplication in \( \mathbb{N}^0 \ltimes S \) is defined by the following rule:

\[
(k, s) \cdot (l, t) = (k + l, (s \theta^l)t).
\]

Define for \( m \in \mathbb{N}^0 \) and \( (l, s) \in \mathbb{N}^0 \ltimes S \)

\[
m \cdot (l, s) = (g - m, s \theta^{g-l}) \quad \text{and} \quad m(l, s) = g - l
\]

where \( g \) is greater of \( m \) and \( l \). Then \( (\mathbb{N}^0 \ltimes S) \times \mathbb{N}^0 \) is Zappa-Szép product with composition rule

\[
[(k, s), m] \circ [(l, t), n] = [(k - m + g, s \theta^{g-m} t \theta^{g-l}), n - l + g],
\]

where again \( g \) is greater of \( m \) and \( l \).
Theorem 2 Let $S$ be a special $\tilde{D}_E$-simple restriction monoid. Then $M = L \Join \tilde{R}_E^1$ is a Zappa-Szép product of $L$ and $\tilde{R}_E^1$ under the actions defined by

$$r \cdot l = d \text{ where } d \in L \text{ and } d^+ = (rl)^+$$

and

$$r'^l = d^\circ rl \text{ where } d \in L \text{ and } d^+ = (rl)^+$$

for $l \in L$ and $r \in \tilde{R}_E^1$. Further $S \cong M$. 
We explain these actions with the help of an egg box picture.

<table>
<thead>
<tr>
<th></th>
<th>r</th>
<th>( r' = d \circ rl )</th>
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<tr>
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<tr>
<td>l</td>
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<tr>
<td>( r \cdot l = d )</td>
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<td>( rl )</td>
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Special $\tilde{D}_E$-simple restriction monoids and Zappa-Szép products

**Theorem 3** Let $S$ be a special $\tilde{D}_E$-simple restriction monoid. Then $Z = \tilde{H}_E^1 \rtimes R$ is a Zappa-Szép product isomorphic to $\tilde{R}_E^1$ under the action of $R$ on $\tilde{H}_E^1$ defined by

$$r \cdot h = rht^*$$

where $t^* = (rh)^*$ and $t \in R$.

and action of $\tilde{H}_E^1$ on $R$ by

$$r^h = t$$

where $t^* = (rh)^*$ and $t \in R$.

Now we see that if $\tilde{H}_E$ is a congruence, then for $r \in R$ and $h \in \tilde{H}_E^1$

$$rh\tilde{H}_E r1 = r$$

and thus $r^h = r$, so that $Z$ becomes a semidirect product.
Kunze showed that if $S$ is a monoid and $\mathbb{N}$ is the set of natural numbers under addition, then a semidirect product $\mathbb{N}^0 \ltimes S$ can be formed under the multiplication,

$$(k, s)(l, t) = (k + l, (s^l)t).$$

Now we see that

$$L_1 = \{(l, s, 0) : l \in \mathbb{N}^0, s \in S\},$$

so that if we put

$$L = \{(l, e, 0) : l \in \mathbb{N}^0\} \cong \mathbb{N}^0,$$

then $L$ is submonoid $\tilde{H}_E$-transversal of $L_1$. Further,

$$\tilde{H}_1 = \{(0, s, 0) : s \in S\}.$$
For \((l, e, 0) \in L\) and \((0, s, 0) \in \tilde{H}_1\),

\[
(0, s, 0)^{(l, e, 0)} = (l, e, 0)^{-1}(0, s, 0)(l, e, 0) = (0, s\theta^l, 0) \in \tilde{H}_1.
\]

Thus \(L \ltimes \tilde{H}_1\) is semidirect product under multiplication defined by

\[
((k, e, 0), (0, s, 0)) ((l, e, 0), (0, t, 0)) = ((k + l, e, 0), (0, s\theta^l t, 0)).
\]
Applications to bisimple inverse monoids

We specialise Theorem 2 and Theorem 3 to obtain corresponding results for bisimple inverse monoids.

Example

The bicyclic semigroup $B$ is the Zappa-Szép product of $L = L_1$ and $R = R_1$, where

$L = \{(m, 0) : m \in \mathbb{N}^0\} \cong \mathbb{N}^0$

$R = \{(0, n) : n \in \mathbb{N}^0\} \cong \mathbb{N}^0$

under the actions of $R$ on $L$ and $L$ on $R$ defined respectively as:

$(0, m) \cdot (n, 0) = (\max(m, n) - m, 0)$

and

$(0, m)^{(n, 0)} = (0, \max(m, n) - n)$. 