K-Theory of Inverse Semigroups

Alistair R. Wallis
Supervisor: Mark V. Lawson

Heriot-Watt University

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Dualities in mathematics:

- Order structures and discrete spaces (Stone duality)
- Locally compact Hausdorff spaces and commutative $C^*$-algebras (Gelfand representation theorem)

Also, (von Neumann) regular rings similar to regular semigroups
Paterson (1980’s) and Renault (1980) generalised this to deep connections between 3 different ”discrete” mathematical structures:

- Inverse semigroups (generalised order structures)
- Topological groupoids (generalised discrete spaces)
- C*-algebras
Background

Some successful applications:

- Topological $K$-theory and operator / algebraic $K$-theory (Serre-Swan theorem)
- Module theory for rings (Dedekind + others) and act theory for monoids
- Morita equivalence of semigroups (Knauer, Talwar) vs. Morita equivalence for rings (Morita)
- Morita equivalence for inverse semigroups (Afara, Funk, Laan, Lawson, Steinberg) vs. Morita equivalence for $C^*$-algebras (Rieffel + others)
Examples

- Polycyclic / Cuntz monoid / Cuntz groupoid / Cuntz algebra
- Graph inverse semigroups / Cuntz-Krieger semigroups / Cuntz-Krieger groupoid / Cuntz-Krieger algebra
- Boolean inverse monoids / Boolean groupoids
- Tiling semigroups / tiling groupoids / tiling $C^*$-algebras
Theorem
Let $S$ be a commutative semigroup. Then there is a unique (up to isomorphism) commutative group $G = \mathcal{G}(S)$, called the Grothendieck group, and a homomorphism $\phi : S \to G$, such that for any commutative group $H$ and homomorphism $\psi : S \to H$, there is a unique homomorphism $\theta : G \to H$ with $\psi = \theta \circ \phi$. 
Algebraic $K$-theory

- $R$ - ring
- $\text{Proj}_R$ - finitely generated projective modules of $R$.
- $(\text{Proj}_R, \oplus)$ is a commutative monoid.
- Define

\[ K_0(R) = G(\text{Proj}_R). \]

- If $X$ is a compact Hausdorff space and $C(X)$ is the ring of $\mathbb{F}$-valued continuous functions on $X$ then

\[ K_0^0(X) \cong K_0(C(X)). \]
Idempotent matrices

- Let $M(R)$ denote the set of $\mathbb{N}$ by $\mathbb{N}$ matrices over $R$ with finitely many non-zero entries.
- Idempotent matrices correspond to projective modules.
- Say idempotent matrices $E, F \in M(R)$ are similar and write $E \sim F$ if $E = XY$ and $F = YX$ where $X, Y \in M(R)$.

Proposition

Idempotent matrices $E$ and $F$ define the same projective module if and only if $E \sim F$. 
Idempotent matrices

- Denote the set of idempotent matrices by $\text{Idem}(R)$ and define a binary operation on $\text{Idem}(R)/\sim$ by

$$[E] + [F] = [E' + F'],$$

where if a row in $E'$ has non-zero entries then that row in $F'$ has entries only zeros, similarly for columns of $E'$, and for rows and columns of $F'$, and such that $E' \sim E$ and $F' \sim F$.

**Theorem**

This is a well-defined operation and the monoids $\text{Idem}(R)/\sim$ and $\text{Proj}_R$ are isomorphic.

- This gives us an alternative way of viewing $K_0(R)$:

$$K_0(R) = \mathcal{G}(\text{Idem}(R)/\sim).$$
$K$-theory of inverse semigroups

- Idea: want to define $K_0(S)$ for $S$ an inverse semigroup.
- Need to restrict the class of inverse semigroups - will not be a problem.
- Give definition in terms of projective modules and definition in terms of idempotent matrices.
- Want $K_0(S) \cong K_0(C(S))$, where $C(S)$ is some $C^*$-algebra associated to $S$. 

Some inverse semigroup theory

- An inverse semigroup is a semigroup $S$ such that for every element $s \in S$ there exists a unique element $s^{-1} \in S$ with $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$ (without uniqueness, we have a regular semigroup).
- A regular semigroup is inverse if and only if its idempotents commute.
- Natural partial order (NPO): $s \leq t$ iff $s = ts^{-1}s$.
- Remark: the set of idempotents form a meet semilattice under the operation $e \wedge f = ef$. 
Orthogonally complete inverse semigroups

- Firstly, we will assume our inverse semigroup $S$ has a zero $(0s = s0 = 0)$.
- Next, we want our inverse semigroup to be sufficiently ring like, namely we require *orthogonal completeness* - this will not be a problem as every inverse semigroup with 0 has an orthogonal completion and the examples we are interested in are orthogonally complete.
- Elements $s, t \in S$ are *orthogonal*, written $s \perp t$, if $st^{-1} = s^{-1}t = 0$.
- $S$ is *orthogonally complete* if
  1. $s \perp t$ implies there exists $s \lor t$
  2. $s \perp t$ implies $u(s \lor t) = us \lor ut$ and $(s \lor t)u = su \lor tu$. 
Throughout what follows $S$ will be an orthogonally complete inverse semigroup.

A matrix $A$ with entries in $S$ is said to be a rook matrix if it satisfies the following conditions:

1. (RM1): If $a$ and $b$ lie in the same row of $A$ then $a^{-1}b = 0$.
2. (RM2): If $a$ and $b$ lie in the same column of $A$ then $ab^{-1} = 0$.

$R(S) =$ all finite-dimensional rook matrices

$M_n(S) =$ all $n \times n$ matrices

$M_\omega(S) = \mathbb{N} \times \mathbb{N}$ rook matrices with finitely many non-zero entries.
Facts about rook matrices

▶ $R(S)$ is an inverse semigroupoid.
▶ $M_n(S)$ and $M_\omega(S)$ are orthogonally complete inverse semigroups.
▶ Let

$$A(S) = E(M_\omega(S))/\mathcal{D}.$$  

▶ Define $[E] + [F] = [E' \lor F'].$
▶ We will define

$$K(S) = G(A(S)).$$  

▶ $S \mapsto M_\omega(S)$ and $S \mapsto K(S)$ have functorial properties.
Pointed étale sets

A pointed étale set is a set $X$ together with a right action of $S$ on $X$, a map $p : X \rightarrow E(S)$ and a distinguished element $0$ satisfying the following:

- $x \cdot p(x) = x$.
- $p(x \cdot s) = s^{-1}p(x)s$.
- $p(0_X) = 0$ and if $p(x) = 0$ then $x = 0_X$.
- $0_X \cdot s = 0_X$ for all $s \in S$.
- $x \cdot 0 = 0_X$ for all $x \in X$.

Define a partial order on $X$: $x \leq y$ iff $x = y \cdot p(x)$.

Define $x \perp y$ if $p(x)p(y) = 0$ and say that $x$ and $y$ are orthogonal.

We will say elements $x, y \in X$ are strongly orthogonal if $x \perp y$, $\exists x \lor y$ and $p(x) \lor p(y) = p(x \lor y)$.
Premodules and modules

A *premodule* is a pointed étale set such that

- If \( x, y \in X \) are strongly orthogonal then for all \( s \in S \) we have \( x \cdot s \) and \( y \cdot s \) are strongly orthogonal and
  \[ (x \lor y) \cdot s = (x \cdot s) \lor (y \cdot s). \]
- If \( s, t \in S \) are orthogonal then \( x \cdot s \) and \( x \cdot t \) are strongly orthogonal for all \( x \in X \).

A *module* is a pointed étale set such that

- If \( x \perp y \) then \( \exists x \lor y \) and \( p(x \lor y) = p(x) \lor p(y) \).
- If \( x \perp y \) then \( (x \lor y) \cdot s = x \cdot s \lor y \cdot s \).
Examples

- 0 is a module with $0 \cdot s = 0$ for all $s \in S$ (initial object in category).
- $eS$ is a premodule with $es \cdot t = est$ and $p(es) = s^{-1}es$.
- $S$ itself is a premodule with $s \cdot t = st$ and $p(s) = s^{-1}s$.
- In fact, every right ideal is a premodule.
Categories

▶ We will define premodule morphisms and module morphisms $f : (X, p) \to (Y, q)$ to be structure preserving maps between, respectively, premodules and modules.

▶ Note that we require $q(f(x)) = p(x)$.

▶ We denote the category of premodules of $S$ by $\text{Premod}_S$ and modules by $\text{Mod}_S$.

▶ Monics in $\text{Premod}_S$ and $\text{Mod}_S$ are injective and epics in $\text{Mod}_S$ are surjective.

▶ $\text{Mod}_S$ is cocomplete.

Proposition

There is a functor $\text{Premod}_S \to \text{Mod}_S$, $X \mapsto X^\#$, which is left adjoint to the forgetful functor.
Can define coproduct in $\textbf{Mod}_S$ for $(X, p), (Y, q)$ by

$$X \bigoplus Y = \{(x, y) \in X \times Y | p(x)q(y) = 0\}$$

with

$$(p \oplus q)(x, y) = p(x) \lor q(y)$$

and

$$(x, y) \cdot s = (x \cdot s, y \cdot s).$$
Projective modules

- A projective module $P$ is one such that for all morphisms $f : P \rightarrow Y$ and epics $g : X \rightarrow Y$ there is a map $h : P \rightarrow X$ with $gh = f$.
- If $P_1, P_2$ projective then $P_1 \oplus P_2$ is projective.
- $(eS)^\#$ is projective.
- Denote by $\text{Proj}_S$ the category of modules $X$ with

$$X \cong \bigoplus_{i=1}^{m} (e_i S)^\#.$$
Theorem
Let $e = (e_1, \ldots, e_m)$, $f = (f_1, \ldots, f_n)$ and $\Delta(e), \Delta(f)$ be the associated diagonal matrices in $M_\omega(S)$. Then

$$\bigoplus_{i=1}^{m} (e_i S)^\# \cong \bigoplus_{i=1}^{n} (f_i S)^\#$$

if, and only if,

$$\Delta(e) \mathcal{D} \Delta(f).$$

Corollary

$$K(S) = G(\text{Proj}_S).$$
Can define states and traces on $S$

- If $S$ commutative, then $K(S) \cong K(E(S))$.

- If $S$ commutative or nice then can form tensor products of matrices and modules - sometimes gives a ring structure on $K(S)$. 
Examples

- Symmetric inverse monoids:
  \[ K(I_n) = \mathbb{Z}. \]

- (Unital) Boolean algebras:
  \[ K(A) = K^0(S(A)). \]

- Cuntz-Krieger semigroups:
  \[ K(CK_G) = K^0(O_G). \]
Thank you for listening