Products in Transformation Semigroups: work with the late John M. Howie

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From 1998, John Howie, Nik Ruskuc and I wrote a series of papers stemming from Howie’s early efforts [6] on the subsemigroup $\langle E \rangle$ of $T_X$ generated by the set of idempotents $E$. This blossomed in two directions:
1. the description of products of special sets in terms of parameters of the mappings involved and
2. ‘rank’ theorems where smallest sets of additional generators were identified to produce all of some type of transformation semigroup.

The first parameters, sets and relations that arose were:
The shift of $\alpha$, $S(\alpha) = \{x \in X : x\alpha \neq x\}$; the fix of $\alpha$, $F(\alpha) = X \setminus S(\alpha)$;
the defect of $\alpha$, $D(\alpha) = X \setminus X\alpha$,
$\ker \alpha = \{(x, y) \in X \times X : x\alpha = y\alpha\}$.
The cardinals of these sets are denoted $s(\alpha)$, $f(\alpha)$, and $d(\alpha)$ while $\text{Ker } \alpha$ is the partition of $X$ corresponding to $\ker \alpha$.
Another cardinal featured, the collapse of $\alpha$: $c(\alpha) = |X \setminus T_\alpha|$, where $T_\alpha$ is any transversal of Ker $\alpha$. 
In his seminal 1966 paper John proved that $\langle E \rangle = F^1 \cup Q$ where
$F = \{ \alpha \in T_X : d(\alpha) > 0 \text{ and } s(\alpha) < \aleph_0 \}$,
$Q = \{ \alpha \in T_X : s(\alpha) = d(\alpha) = c(\alpha) \geq \aleph_0 \}$
$Q$ is the set of balanced elements. In the paper [3] however arose the set of semi-balanced elements $B = \{ \alpha \in T_X : d(\alpha) = c(\alpha) \}$ as did the sets $C = \{ \alpha \in T_X : d(\alpha) \leq c(\alpha) \}$ and $D = \{ \alpha \in T_X : d(\alpha) \geq c(\alpha) \}$. The main results of the paper are summarised in the following table.
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Semigroup Table of products of sets in T
T = Full transformation semigroup, S = Surjections, I = Injections, J = top J-class, G = full symmetric group, B = Semi-Balanced elements, C = Collapse majorises Defect, D = Defect majorises Collapse

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It follows from the table that $\langle E \cup G \rangle = B$ and we proved that

**Theorem**

[3, Theorem 5.7] If $\langle E \cup G \cup M \rangle = T_X$ then $M$ contains a proper surjection and a proper injection. Moreover there is a solution to this equation where $|M| = 2$.

In [4] we solved more rank problems where we defined the (relative) rank of a semigroup $S$ modulo a set $A \subseteq S$ to be the least cardinal $r$ such that $r = |B|$ where $B \subseteq S$ such that $\langle A \cup B \rangle = S$. This is denoted by $\text{rank}(S : A)$. We proved:

**Theorem**

[4, Theorems 4.1 and 5.1] For an infinite base set $X$, $\text{rank}(T_X : S_X) = \text{rank}(T_X : E_X) = 2$. 
In [3] we included descriptions of products of $E$ with arbitrary $\mathcal{H}$-, $\mathcal{R}$- and $\mathcal{L}$- classes of $T_X$. An $\mathcal{L}$-class $L = L_\alpha$, is determined by $Y \subseteq X$, the common range of members of $L$; an $\mathcal{R}$-class $R_\alpha$ is determined by a common partition $\Pi = \text{Ker} \alpha$ of $X$, and so the $\mathcal{H}$-class $H = L \cap R$ is determined by $(\Pi, Y)$ so that $H = \{ \alpha \in T_n : \text{Ker} \alpha = \Pi \text{ and } X \alpha = Y \}$. (Recall that $\mathcal{D} = \mathcal{I}$ in $T_X$ with $(\alpha, \beta) \in \mathcal{D}$ if and only if $|X \alpha| = |X \beta|$.)

**Definitions**

The $Y$-kernel of $\alpha$ is $\text{Ker}_Y(\alpha) = \{ K \in \text{Ker} \alpha : K \alpha \in Y \}$. A $Y$-transversal of $\alpha$ is a transversal of $\text{Ker}_Y \alpha$. The *relative defect of $\alpha$ is* $D_Y(\alpha) = Y \cap \overline{X \alpha}$ with cardinal $d_Y(\alpha)$. The *relative collapse of $\alpha$ with respect to $Y$ is* the cardinal $c_Y(\alpha) = |X \setminus \tau|$, where $\tau$ is some $Y$-transversal. The *supplement of a partition $\Pi$ of $X$ with respect to $\tau \subseteq X$ is denoted by $s_\tau(\Pi)$ and is the cardinal of the set

$$S_\tau(\Pi) = \{ P \in \Pi : P \cap \tau = \emptyset \}.$$ 

The *excess of $\alpha$ with respect to $Y$, denoted by* $e_Y(\alpha) = |X \alpha \cap \overline{Y}|$. 

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Definitions

Let $\alpha \in T_X$, $Y \subseteq X$ and $\Pi$ be a partition of $X$.

Condition C: there exists a $Y$-transversal $\tau$ of $\alpha$ that is a partial transversal of $\Pi$ and is such that $s_\tau(\Pi) = d_Y(\alpha)$.

Theorem

(from Thm 2.1 of [3])
(a) $\alpha \in EH$ iff $X\alpha \subseteq Y$ and $\alpha$, $Y$ and $\Pi$ satisfy (C).
(b) $\alpha \in HE$ iff $\ker \alpha \leq \Pi$ and $\alpha$, $Y$ and $\Pi$ satisfy (C).

(Thms 2.5, 2.7 of [3])
(a) $\alpha \in LE$ iff $c_Y(\alpha) \geq d_Y(\alpha) \geq e_Y(\alpha)$; $\alpha \in EL$ if additionally $e_Y(\alpha) = 0$.
(b) $\alpha \in ER$ iff $s_\tau(\Pi) \leq d(\alpha)$ for some transversal $\tau$ of $\ker \alpha$ that is a partial transversal of $\Pi$; $\alpha \in RE$ if additionally $\ker \alpha \leq \Pi$. 
**Theorem**

(Thm. 4.1 of [2]) Let $L$ (resp. $R$) be an $L$-class (resp. $R$-class) contained in $J$ other than $S$ (resp. other than $I$). Then $L$ (resp. $R$) contains an $H$-class $H$ such that none of the sets $HE$, $LE$, $EH$, $EL$, (resp. $EH$, $ER$, $HE$, and $RE$) are subsemigroups of $T_X$.

**Definitions**

(See [1]) Let $\leq$ be the *natural partial order* on a semigroup $S$, whereby $a \leq b$ iff there exists $s, t \in S^1$ such that $sb = sa = a = at = bt$; if $S$ is regular we may take $s, t \in E(S)$. Let $S_Y = \{\alpha \in T_X : X\alpha \subseteq Y\}$ and $S_\Pi$ consists of all $\alpha$ such that $\text{Ker } \alpha$ has a transversal $\tau$ that is a partial transversal of $\Pi$; $S_Y$ and $S_\Pi$ are respectively a left and a right ideal of $T_X$. 
Theorem

Let \( L = L_\alpha \) be an \( L \)-class and \( R_\alpha \) be an \( R \)-class, both not contained in \( J \). Then
\[
(a) \quad EL = L_\downarrow = T\alpha \subset LE = S_Y;
(b) \quad RE = R_\downarrow = \alpha T \subset ER = S_\Pi.
\]

Corollary

Let \( X \) be finite. Then
\[
(a) \quad HE = RE = \alpha T;
(b) \quad EH = S_Y \cap S_\Pi \text{ is a subsemigroup of } T \text{ and a union of } H \text{-classes of } T_X.
\]
In general, $EH \subseteq S_Y \cap S_\Pi$ and conversely, if $\alpha \in S_Y \cap S_\Pi$ then for any transversal $\tau$ of $\text{Ker}\alpha$ that is a partial transversal of $\Pi$:

$$|\tau| + s_\tau(\Pi) = |X\alpha| + d_Y(\alpha)$$

since $|\tau| = |X\alpha|$, in the finite case we have simply by subtraction that $s_\tau(\Pi) = d_Y(\alpha)$, which ensures that $\alpha \in EH$ by Condition C. Which $\mathcal{H}$-classes comprise $S = EH$?
Theorem

[Thm. 2.2 of [2]] Let $H_1$ be defined by $(Y_1, \Pi_1)$ with $|Y_1| = t$. Then $H_1 \subseteq EH$ iff $Y_1 \subseteq Y$ and $\Pi_1$ satisfies Hall’s Condition with respect to $\Pi$.

Hall’s Condition: no union of $k$ classes of $\Pi$ contains more than $k$ classes of $\Pi_1$ ($1 \leq k \leq t - 1$).

Theorem

[Thm. 2.6 of [2]] Let $\alpha, \beta \in S = EH$ be distinct mappings. Then
(a) $L_\beta \leq L_\alpha$ iff $X\beta \subseteq Y\alpha$; $\alpha L \beta$ iff $X\alpha = Y\alpha = X\beta = Y\beta$.
(b) $R_\beta \leq R_\alpha$ iff $\text{Ker} \beta \leq \text{Ker} \alpha$; $\alpha R \beta$ iff $\text{Ker} \alpha = \text{Ker} \beta$.
(c) $\alpha \in \text{Reg}(S)$ iff $X\alpha = Y\alpha$. 
Definitions

A partition \( \Gamma \) of \( X \) that satisfies Hall’s Condition wrt \( \Pi \) is called **regular** wrt \( S = EH \) if \( K \cap Y \neq \emptyset \) for every \( K \in \Gamma \). Let \( S_R \) (resp. \( S_I \)) denote the set of all regular (resp. irregular) partitions of \( X \) wrt \( S \).

Theorem

For each \( \Gamma \in S_I \) there is an irregular \( \mathcal{D} \)-class \( D_\Gamma \) whereby \( \alpha \in D_\Gamma \) iff \( \text{Ker} \alpha = \Gamma \) and \( X\alpha \subseteq Y \) with \( |X\alpha| = |\Gamma| \). In particular, each irregular \( \mathcal{D} \)-class \( D \) is also an \( \mathcal{R} \)-class and each \( \mathcal{L} \)-class within \( D \) is trivial. If \( S \) is not regular (occurs when \( |Y| \geq 2 \) and \( H \neq S_X \)) then there exists irregular members of \( S \) of all ranks \( 2 \leq m \leq |Y| \).

There are \( |Y| \) (non-empty) regular \( \mathcal{D} \)-classes \( D_m \), \( (1 \leq m \leq |Y|) \):

\[
D_m = \{ \alpha : |X\alpha| = m, X\alpha \subseteq Y, \text{Ker} \alpha \text{ a regular partition of } X \text{ wrt } S \}.
\]

Furthermore \( \text{Reg}(S) \) is a right ideal of \( S \).


