The construction of the direct product of two algebraic structures is a relatively basic and well-behaved operation. It is often true, particularly in the class of groups, that algebraic properties such as finite generation, finite presentation, and solubility amongst others are preserved by direct products. Moreover, the converse is also true; each factor in a direct product often inherits such properties. Of course, direct products are not restricted to the class of groups, and comparing and contrasting the effect of the direct product on generalisations of these properties between different classes of algebras is of interest. The situation acquires further complexity for the related subdirect product however, being a subalgebra of the direct product with particular properties. There are examples even in the class of groups for which the aforementioned properties are not preserved when considering subdirect products. A more complex analysis of combinatorial properties for subdirect products is necessary and has been a topic of research in recent years, particularly in the areas of group theory, semigroup theory and universal algebra.

This 5-part lecture series will be entitled “Direct and Subdirect Products in Groups, Semigroups and Algebras”. The intended audience for this series will be postgraduates entering a PhD in any form of combinatorial algebra, though indeed anyone with a familiarity from undergraduate courses in this area may also benefit, particularly postgraduate students at later stages. In summary, the aim of the lecture series will be to introduce the algebraic construction of the direct product and the related notion of subdirect products to the audience, in the settings of group theory (which should already be familiar from undergraduate courses), semigroup theory and more widely in universal algebra. Starting with the familiar case of groups, it will discuss both the abstract motivation and uses of these constructions. We will then emphasise abstraction to the related areas of semigroup theory and universal algebra. In particular, we will discuss the effects of these constructions on structural properties in all of these algebraic structures. We will start with the properties familiar from undergraduate courses such as finite generation, nilpotency and solubility, and then move to less well-known properties stemming from more general combinatorial algebra. In the latter parts of the series, we will move towards classic and recent research questions concerning direct and subdirect products, with the intention of highlighting some active research topics to the audience; providing potential research directions for postgraduates; and promoting the area of Semigroup theory and research interests represented within NBSAN.

The lecture series will start by defining preliminaries such as the direct product for two groups as relatively familiar concepts for the audience. Elementary properties will be shown or at least stated, including the standard criterion for direct decomposability. The subdirect product will be further defined rather than considering it as a construction technique, and expressed as a subgroup of the direct product for which projections onto the factors are surjective. Some motivation for considering subdirect products will be explored, such as seeing every group is reducible into a subdirect product of subdirectly irreducible groups, and discussing their role in representing small perfect groups. Fiber products will be defined, and that every subdirect product of two groups can be realised as a fiber product will be covered (known as Goursat’s lemma).

Having explored some hopefully familiar concepts in group theory, we will observe that direct and subdirect products are of course not restricted to groups, and can be defined for any type of algebraic structure. The lecture series will hence move to introducing monoids and semigroups as larger such algebraic classes, of course containing groups. Basic concepts and definitions in semigroup theory will be introduced when relevant, particularly when related to direct products and subdirect products. This will allow us to compare and contrast the behaviour between groups and semigroups in the direct and subdirect product. For example, not every direct product of two semigroups contains a subsemigroup isomorphic to each factor. We will also briefly explore the definitions of the direct product and subdirect product in the setting of universal algebra, particularly noting that the role of normal subgroups will be played by congruences.

With the theoretical foundations having been established, the remainder of the lecture series will cover research themes relating to direct or subdirect products. The first theme will be that of substructures in direct products. In this theme, we will discuss material related to the questions of what isomorphic substructures are contained in direct products, how many are there, what types of subdirect products can be found in a direct product, how do groups and semigroups differ in relation to these questions, amongst others. The remaining themes will explore the properties P that are reserved under direct products, for which we will consider the following main question: do two algebras have a property P if and only if the direct product also has property P? As one theme in the case of groups, we will define and discuss the positive case for P the properties of finite generation and finite presentation. We will then move to show the results from groups are not true in general for presentations of semigroups. We will also discuss recent work on this theme in both the settings of universal algebra and semigroup theory for subdirect products. The final theme will discuss further finitary properties in the same context, such as residual finiteness, nilpotency, complete separability for semigroups amongst others. This theme will pay particular attention to the recent research work of various members of NBSAN.