Tropical Representations and Identities of Plactic Monoids

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The tropical semifield

Definition

Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and consider two binary operations defined by:

$$x \oplus y := \max(x, y), \quad x \otimes y := x + y.$$

Properties

$\mathbb{T}$ *is an* **idempotent semifield***:*

- $(\mathbb{T}, \oplus)$ *is a commutative monoid with identity* $-\infty$*;*
- $-\infty$ *is a zero element for* $\otimes$*;*
- $(\mathbb{T} \setminus \{-\infty\}, \otimes)$ *is an abelian group with identity* 0*;*
- $\otimes$ *distributes over* $\oplus$*;*
- $\oplus$ *is idempotent:* $x \oplus x = x$

- Note that 0 is 'one' and $-\infty$ is 'zero'.
- We also have $x \oplus y$ is either x or y.

What is ‘tropical maths’? And why is it interesting?

Definition

Tropical algebra or **max-plus algebra** is linear algebra where the base field is replaced by the tropical semiring. **Tropical geometry** is (roughly!) algebraic geometry where the base field is replaced by the tropical semiring.

Tropical methods have applications in . . .

- Combinatorial Optimisation
- Discrete Event Systems
- Phylogenetics
- Numerical Analysis
- Economics
- (Mostly Enumerative) Algebraic Geometry
- Formal Languages and Automata
- **Semigroup Theory** (carrier for representations)

Tropical Matrix Semigroups

Definition

$M_n(\mathbb{T})$ is the semigroup of $n \times n$ matrices over $\mathbb{T}$, under the natural matrix multiplication induced by $\oplus$ and $\otimes$.

Definition

$UT_n(\mathbb{T})$ is the subsemigroup of upper triangular matrices.

- Studied implicitly for 50+ years with many interesting specific results (e.g. Gaubert, Cohen-Gaubert-Quadrat, d'Alessandro-Pasku).
- Since about 2008, systematic structural study using the tools of semigroup theory
(J.& Kambites + Gould, Hollings, Izhakian, Naz, Taylor, Wilding, ...).

Example

$$\begin{pmatrix} 2 & 1 \\ -\infty & 19 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 \\ -20 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 23 \end{pmatrix}$$

Tropical Representations

Definition

A semigroup S admits a tropical representation if there is a morphism from S to $M_n(\mathbb{T})$ for some n.
Say that the representation is upper triangular if the image lies in $UT_n(\mathbb{T})$.
Say that the representation is faithful if the morphism is injective.

Example

- Every finite semigroup admits a faithful tropical represention.
- d'Alessandro and Pasku, 2003: Finitely generated subsemigroups of $M_n(\mathbb{T})$ have polynomial growth. (So free monoids of rank $r \geq 2$ do not admit faithful tropical representations.)
- Izhakian and Margolis, 2010: The bicyclic monoid admits a faithful upper triangular tropical representation.

Semigroup Identities

A **semigroup identity** is a pair of non-empty words, usually written $u = v$ over some alphabet Σ.

A semigroup S **satisfies** the identity $u = v$ if every morphism from the free semigroup Σ^+ to S sends u and v to the same place.

(In other words, if u and v evaluate to the same element for every substitution of elements in S for the letters in Σ.)

Example

A semigroup satisfies ...

- ... $AB = BA$ if and only if it is commutative;
- ... $A^2 = A$ if and only if it is idempotent;
- ... $AB = A$ if and only if it is a left-zero semigroup.

Identities for upper triangular tropical matrices

Let $u(A,B) = ABBA\,AB\,ABBA$ and $v(A,B) = ABBA\,BA\,ABBA$.

Theorem (Izhakian & Margolis 2010)

$UT_2(\mathbb{T})$ and $M_2(\mathbb{T})$ satisfy (non-trivial) semigroup identities.

On the identities constructed...

$UT_2(\mathbb{T})$ satisfies $u(A,B) = v(A,B)$; each word has length 10
$M_2(\mathbb{T})$ satisfies $u(A^2,B^2) = v(A^2,B^2)$; each word has length 20.

Theorem (Izhakian 2013–16, Okniński 2015, Taylor 2016)

$UT_n(\mathbb{T})$ satisfies a semigroup identity for every n.

Okniński's construction

Set $u_1 = u(A,B)$ and $v_1 = v(A,B)$ and for $j \geq 1$ set

$$u_{j+1} = u(u_j, v_j), \quad v_{j+1} = v(u_j, v_j).$$

Then $UT_n(\mathbb{T})$ satisfies $u_{n-1} = v_{n-1}$; each word has length 10^{n-1}

Identities and the bicyclic monoid

Let $u(A, B) = ABBA\,AB\,ABBA$ and $v(A, B) = ABBA\,BA\,ABBA$.
Adian showed that the identity $u(A, B) = v(A, B)$ is satisfied by the bicyclic monoid.

Theorem (Izhakian & Margolis 2010)

(i) $UT_2(\mathbb{T})$ satisfies Adian's identity.
(ii) The bicyclic monoid has a faithful representation in $UT_2(\mathbb{T})$, and hence satisfies all identities satisfied in $UT_2(\mathbb{T})$.

Question (Izhakian & Margolis 2010)

Does the bicyclic monoid satisfy exactly the same identities as $UT_2(\mathbb{T})$?

Theorem (Daviaud, J. & Kambites 2018)

- *$UT_2(\mathbb{T})$ satisfies exactly the same identities as the bicyclic monoid.*
- *For each n there is an efficient algorithm to check whether a given identity is satisfied in $UT_n(\mathbb{T})$.*

Identities for tropical matrix semigroups

Theorem (Izhakian & Merlet 2018, building on ideas of Shitov)

$M_n(\mathbb{T})$ satisfies a semigroup identity for every n.

On the identities constructed...

- *Shitov showed that $M_3(\mathbb{T})$ satisfies a non-trivial identity; the length of the words involved in his original construction is over 1 million.*
- *Izhakian and Merlet have shown that, for each n, $M_n(\mathbb{T})$ satisfies a non-trivial identity. In the case $n = 3$, the words have length around $19,000$.*

Philosophy

If you can find a tropical matrix representation of your favourite semigroup, then you can conclude that this semigroup satisfies a non-trivial identity...

Plactic Monoids

The **plactic monoid** $\mathbb{P}_n$ of rank n is the monoid generated by $\{1, 2, \ldots, n\}$ $(= [n])$ subject to the **Knuth relations**:

$$bca = bac \quad (a < b \leq c) \qquad\qquad acb = cab \quad (a \leq b < c)$$

Elements are in bijective correspondence (via row reading or column reading) with **semistandard Young tableaux** over $[n]$:

4	4		
2	3	4	
1	2	3	3

$= 442341233 = 421432433 = \cdots$

4			
2	3	4	4
1	2	3	3

$= 423441233 = 421324343 = \cdots$

(Entries in each column strictly decreasing, entries in each row weakly increasing, row lengths weakly increasing.)

Plactic Monoids...

- ...arise from the study of **Schensted's algorithm** (1961) which constructs tableaux from words.
- ...were (first?) discovered by Knuth (1970).
- ...were named ("*plaxique*") and extensively studied by Lascoux and Schützenberger (1981).
- ...(and their algebras) have many applications in algebraic combinatorics and representation theory.
- ...are $\mathcal{J}$-trivial.
- ...have polynomial growth of degree $\frac{n(n+1)}{2}$.
- ...admit finite complete rewriting systems and biautomatic structures (Cain, Gray & Malheiro 2015).
- ...

Identities for plactic monoids

Question

Does $\mathbb{P}_n$ satisfy a non-trivial semigroup identity?

- "Yes" when $n \leq 3$ (Kubat & Okniński 2013)
- Conjectured "yes" for all finite n (Kubat & Okniński 2013)
- "No" when n infinite (Cain, Klein, Kubat, Malheiro & Okniński 2017)
- Again conjectured "yes" for all finite n (Cain & Malheiro 2018)
- Preprint of Okniński (2019) on $n \geq 4$ withdrawn.

Corresponding answer is...

- ... "yes" for for Chinese monoids (consequence of Jaszuńska and Okniński 2011)
- ... "yes" for hypoplactic, sylvester, Baxter, stalactic and taiga monoids (Cain & Malheiro 2018)
- ... "yes" for right patience sorting monoids and "no" for left patience sorting monoids (Cain, Malheiro & F. M. Silva 2018)

Tropical representations of plactic monoids

Theorem (Izhakian 2017)

The plactic monoid $\mathbb{P}_3$ has a faithful representation in $UT_3(\mathbb{T}) \times UT_3(\mathbb{T})$.

Cain, Klein, Kubat, Malheiro & Okniński 2017

Alternative faithful representation for $\mathbb{P}_3$.

Question (Izhakian 2017)

Does each $\mathbb{P}_n$ have a **faithful** *tropical representation?*

Both the above representations generalise naturally to higher rank but do **not** remain faithful. e.g. in $\mathbb{P}_4$ they do not separate:

4	4		
2	3	4	
1	2	3	3

and

4			
2	3	4	4
1	2	3	3

Tropical representations of plactic monoids

Theorem (J. & Kambites 2019)

For every finite n, $\mathbb{P}_n$ has a faithful upper triangular tropical representation.

Corollary

Every finite rank plactic monoid satisfies a semigroup identity.

In general the size of our representation is of order 2^n but ...

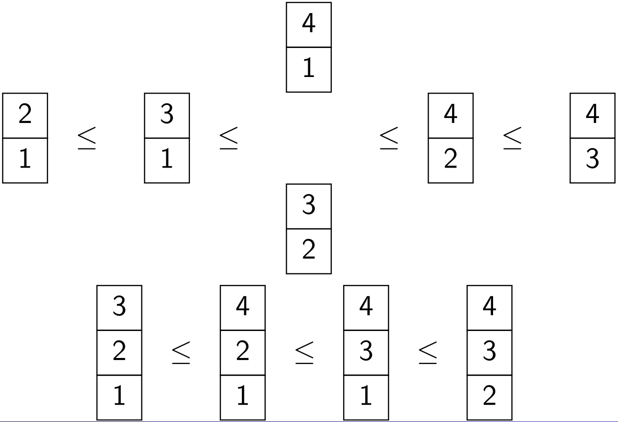
Theorem (J. & Kambites. 2019)

$\mathbb{P}_n$ satisfies all identities satisfied by $UT_d(\mathbb{T})$ where $d = \lfloor \frac{n^2}{4} + 1 \rfloor$

To prove this we use a result of Daviaud, J. & Kambites, 2018.
(Note that $n = 3 \implies d = 3, \quad n = 4 \implies d = 5, \quad n = 5 \implies d = 7$)

Construction of the Representation

- For $\mathbb{P}_n$ we will build $2^n \times 2^n$ matrices, entries indexed by subsets of $[n]$.
- Think of subsets as possible columns of semistandard Young tableaux.
- Define $S \leq T$ if $|S| = |T|$ and column S can appear left of column T.
- For example, with $n = 4$: $\boxed{1} \leq \boxed{2} \leq \boxed{3} \leq \boxed{4}$,



Construction of the Representation

- For $x \in [n]$ define a $2^{[n]} \times 2^{[n]}$ tropical matrix by

$$\rho(x)_{P,Q} = \begin{cases} -\infty & \text{if } P \not\leq Q \\ 1 & \text{if } \exists T \subseteq [n] \text{ with } P \leq T \leq Q \text{ and } x \in T \\ 0 & \text{otherwise.} \end{cases}$$

- Choose an order of rows and columns such that these matrices are upper triangular (by extending $\leq$ to a linear order).
- Extend to a morphism $\rho : [n]^* \to UT_{2^n}(\mathbb{T})$.

We show that...

(i) The map ρ respects the Knuth relations and therefore induces a morphism

$$\rho_n : \mathbb{P}_n \to UT_{2^n}(\mathbb{T}).$$

(ii) The map $\rho_n : \mathbb{P}_n \to UT_{2^n}(\mathbb{T})$ is a faithful representation of $\mathbb{P}_n$.

Identities and Chain Length

So far: $\mathbb{P}_n$ satisfies every semigroup identity satisfied by $UT_N(\mathbb{T})$, $N = 2^n$.

Definition

- Let $\leq$ be a partial order on $[N]$.
- Let d be the length of the longest chain.
- Consider the set of all matrices in $M_N(\mathbb{T})$ such that $i \not\leq j \implies M_{i,j} = -\infty$.
- This is a subsemigroup of $M_N(\mathbb{T})$, called a **chain-structured tropical matrix semigroup** of **chain length** d.

Theorem (Daviaud, J. & Kambites. 2018)

Any chain-structured tropical matrix semigroup of chain length d satisfies the same identities as $UT_d(\mathbb{T})$.

Thus: $\mathbb{P}_n$ satisfies every semigroup identity satisfied by $UT_d(\mathbb{T})$, where $d = \lfloor \frac{n^2}{4} + 1 \rfloor$.

Plactic monoids of low rank

- $\mathbb{P}_1$ satisfies exactly the same identities as a free commutative monoid and hence as $UT_1(\mathbb{T}) = \mathbb{R} \cup \{-\infty\}$.
- $\mathbb{P}_2$ satisfies exactly the same identities as the bicyclic monoid and hence as $UT_2(\mathbb{T})$.

Theorem (Izhakian 2017)

$\mathbb{P}_3$ has a faithful representation in $UT_3(\mathbb{T}) \times UT_3(\mathbb{T})$, ...
... and hence satisfies all identities satisfied in $UT_3(\mathbb{T})$.

Question (Izhakian 2017)

Does $\mathbb{P}_3$ satisfy exactly the same identities as $UT_3(\mathbb{T})$?

Theorem (J. & Kambites 2019)

Yes!

More precisely...

Have seen: $\mathbb{P}_n$ satisfies all identities satisfied by $UT_d(\mathbb{T})$, for $d = \lfloor \frac{n^2}{4} + 1 \rfloor$

Theorem (J. & Kambites 2019)

$UT_n(\mathbb{T})$ satisfies all identities satisfied by $\mathbb{P}_n$.

Corollary

For $n \leq 3$, $\mathbb{P}_n$ satisfies exactly the same identities as $UT_n(\mathbb{T})$ and we can therefore check efficiently whether a given identity is satisfied in $\mathbb{P}_n$.

Question

For larger n:

- *Does $\mathbb{P}_n$ satisfy exactly the same identities as $UT_n(\mathbb{T})$?*
- *Is there an efficient algorithm to decide whether a given identity is satsfied by $\mathbb{P}_n$?*

Varieties

Definition

For S a semigroup let $V(S)$ be the class of semigroups satisfying every identity satisfied by S.

$V(S)$ is a **semigroup variety**
(closed under quotients, subsemigroups and arbitrary direct products).

In total we know....

$$
\begin{aligned}
& V(UT_1(\mathbb{T})) = \mathcal{COM} = V(\mathbb{P}_1) \\
\subsetneq \quad & V(UT_2(\mathbb{T})) = V(\mathcal{B}) = V(\mathcal{FIM}_1) = V(\mathbb{P}_2) \\
\subsetneq \quad & V(UT_3(\mathbb{T})) = V(\mathbb{P}_3) \\
\subsetneq \quad & V(UT_4(\mathbb{T})) \subseteq V(\mathbb{P}_4) \\
\subseteq \quad & V(UT_5(\mathbb{T})) \subseteq V(\mathbb{P}_5) \subseteq V(UT_7(\mathbb{T})) \ \ldots
\end{aligned}
$$