$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

$$+\text{IC}: \quad y(\bar{X}) = \bar{Y}$$
$$y'(\bar{x}) = \bar{z}$$

E & U: Look at p, q, r.

Structure of soln

General soln:

$$y = y_p + c_1 y_1 + c_2 y_2$$

y_p: any soln of full eqn — particular soln.

y_1, y_2: lin. indep., nonzero solns of homog. ODE ($r(x) = 0$) — homog. soln. (complementary fct).

Specific solution:

Apply ICs to determine unique c_1 & c_2

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$$y'' + \underbrace{\frac{1}{x}}_{p(x)} y' - \underbrace{\frac{1}{x^2}}_{q(x)} y = \underbrace{-\frac{1}{x^2}}_{r(x)}$$

$$y = +1 + Ax + B\frac{1}{x}$$ fen. foen.

$$\left.\begin{aligned} y(1) &= \frac{3}{2} \\ y'(1) &= \frac{1}{2} \end{aligned}\right\} \quad y = 1 + \frac{1}{2}x$$

$$y' = \frac{1}{2}$$

$$y'' = 0$$

$$\underbrace{0}_{y''} + \cancel{\frac{1}{x}\frac{1}{2}} - \frac{1}{x^2}\left(1 + \cancel{\frac{1}{2}x}\right) \overset{?}{=} -\frac{1}{x^2}$$

$$-\frac{1}{x^2} \overset{?}{=} -\frac{1}{x^2}$$

Where have we (you!) seen $x = x_P + x_H$ before?

Recall:

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + \alpha\, y_1(x) + \beta\, y_2(x),$$

where:

- α and β are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Compare this to the solution of the system of linear (algebraic) equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{b}$ a given vector of size n.

The general solution $\mathbf{x}$ (another vector of size n) is given by

$$\mathbf{x} = \mathbf{x}_P + \mathbf{x}_H$$

where

- $\mathbf{x}_P$ is a(ny) particular solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$
- $\mathbf{x}_H$ is the *general* solution of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.

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Example

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Note that the matrix is singular, so $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions!

- Transform into "triangular" form

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

  showing that the RHS is consistent. We're left with one equation for three unknowns.

- Set $x_2 = \alpha$ and $x_3 = \beta$, where α and β are arbitrary constants.

- The general solution is: $x_1 = 1 + \alpha$ and, of course, $x_2 = \alpha$ and $x_3 = \beta$.

- Rewrite in vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{x}_P} + \underbrace{\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{x}_H}$$

- Note that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 3.1415 \end{pmatrix}}_{\mathbf{x}'_P} + \underbrace{\alpha' \begin{pmatrix} -42.2 \\ -42.2 \\ 1145.2 \end{pmatrix} + \beta' \begin{pmatrix} 523.2 \\ 523.2 \\ 13.423 \end{pmatrix}}_{\mathbf{x}'_H}$$

  is another (not so pretty) representation of the general solution.

The key features of both solutions are:

- $\mathbf{x}_P$ and $\mathbf{x}'_P$ solve the inhomogeneous equation.
- $\mathbf{x}_H$ and $\mathbf{x}'_H$ "span the null space" of $\mathbf{A}$, i.e. they
  1. satisfy $\mathbf{A}\mathbf{x} = \mathbf{0}$,
  2. are nonzero,
  3. are linearly independent.

"Off the record comment":

In linear algebra it's "easier" to overlook the additional solutions represented by $\mathbf{x}_H$. In an ODE context, the fact that BCs [or ICs] have to be satisfied too, tends to provide an instant "reminder" that just having *a* particular solution of the ODE is not enough to solve the entire IVP/BVP.

How to determine y_P & y_H systematically? Illustrated for:

Constant coefficient ODEs

$$\boxed{\begin{array}{l} y'' + py' + qy = r(x) \\ \text{where } p \,\&\, q \text{ are constants} \end{array}}$$

I Soln. of the homogeneous ODE

$$y'' + py' + qy = 0 \quad (H)$$

Ansatz: $y \sim e^{\lambda x}$ for some constant λ

$$y = Ae^{\lambda x}$$
$$y' = A\lambda e^{\lambda x}$$
$$y'' = A\lambda^2 e^{\lambda x}$$

into ODE:

$$\underbrace{A\lambda^2 e^{\lambda x}}_{y''} + p\underbrace{A\lambda e^{\lambda x}}_{y'} + q\underbrace{Ae^{\lambda x}}_{y} \overset{!}{=} 0$$

$$A e^{\lambda x} \underbrace{(\lambda^2 + p\lambda + q)}_{=0} \overset{!}{=} 0 \quad \forall x$$

$A = 0 \to$ not interesting $\Rightarrow y = 0$.

There are two values of λ: λ_1 & λ_2 which satisfy the "characteristic polynomial"

$$\lambda^2 + p\lambda + q = 0 :$$

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

For these two values of λ the fcts

$$y_1(x) = e^{\lambda_1 x} \quad \& \quad y_2(x) = e^{\lambda_2 x}$$

are nonzero solns. of the homog. ODE (H).

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<u>BUT</u>: possibility of repeated & complex roots.

$\Rightarrow$ 3 cases.

① <u>$p^2 > 4q$: λ_1 & λ_2 are distinct & real</u>

The general soln. of (H) is

$$y = A e^{\lambda_1 x} + B e^{\lambda_2 x}$$

where A & B are constants

② <u>$p^2 = 4q$ Repeated roots</u>

$$\underline{\lambda_1 = \lambda_2 = \lambda = -\frac{p}{2}}$$

Our ansatz produces <u>only</u> one solution to (H):

$$y_1(x) = e^{\lambda x}$$

However $y_2(x) = x e^{\lambda x}$

is another nonzero, lin. indep. soln. of (H).

Proof: Note: $p = -2\lambda$

$q = \frac{1}{4}p^2 = \lambda^2$

$$y = xe^{\lambda x}$$
$$y' = e^{\lambda x}(1+\lambda x)$$
$$y'' = e^{\lambda x}(2+\lambda x)\lambda$$

into ODE:

$$y'' + py' + qy = 0$$

$$e^{\lambda x}\Big(\underbrace{\lambda(2+\lambda x)}_{y''} + \underbrace{p}_{-2\lambda}\underbrace{(1+\lambda x)}_{y'} + \underbrace{q}_{\lambda^2}\underbrace{x}_{y}\Big) \overset{?}{=} 0$$

$$\lambda(2+\lambda x) - 2\lambda(1+\lambda x) + \lambda^2 x \overset{?}{=} 0$$

$$\cancel{2\lambda} + \cancel{\lambda^2 x} - \cancel{2\lambda} - \cancel{2\lambda^2 x} + \cancel{\lambda^2 x} \overset{?}{=} 0$$

✓

$y(x) = Ae^{\lambda x} + Bxe^{\lambda x}$

is the gen. soln. of (H).

③ $p^2 < 4q$: λ_{12} are complex conjugates

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$\lambda_{12} = \underbrace{-\frac{p}{2}}_{\mu \text{ real}} \pm i \underbrace{\sqrt{q - \left(\frac{p}{2}\right)^2}}_{\omega \text{ real}}$$

$$\lambda_{12} = \mu \pm i\omega$$

Gen. soln. is:

$$y = \hat{A} e^{(\mu + i\omega)x} + \hat{B} e^{(\mu - i\omega)x}$$

complex constants

$$y = e^{\mu x}\left(\hat{A} e^{i\omega x} + \hat{B} e^{-i\omega x}\right)$$

Note: $e^{\pm i\omega x} = \cos(\omega x) \pm i \sin(\omega x)$

So, the gen. soln in terms 11
of $e^{\pm i\omega x}$ represents linear combinations of $\cos(\omega x)$ & $\sin(\omega x)$.
Therefore you can also write the gen. soln as

$$y = e^{\mu x}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

where A & B are real constants.

Example:

$$y'' - 3y' + 2y = 0$$

$$y \sim e^{\lambda x}$$

$$e^{\lambda x}\underbrace{(\lambda^2 - 3\lambda + 2)}_{\text{char. poly.}} = 0 \quad \forall x$$

$$\lambda_{12} = +\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2}$$

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$$= +\frac{3}{2} \pm \sqrt{\frac{9-8}{4}}$$

$$= +\frac{3}{2} \pm \frac{1}{2}$$

$\lambda_1 = 2$

$\lambda_2 = 1$

$$\underline{y = Ae^{2x} + Be^{x}}$$

$$y'' + 2y' + y = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0$$

$\lambda_{12} = -1$ repeated

$$\underline{y = Ae^{-x} + Bxe^{-x}}$$

$$y'' + 2y' + 5y = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_{12} = -1 \pm \sqrt{1^2 - 5}$$

$$\lambda_{12} = -1 \pm 2i$$

$$\mu = -1 ; \quad \omega = 2$$

$$y(x) = e^{-x}\left(\hat{A} e^{2ix} + \hat{B} e^{-2ix}\right)$$

$$\underline{\underline{y(x) = e^{-x}\left(A \cos(2x) + B \sin(2x)\right)}}$$

II Particular solns

$$y'' + py' + qy = r(x)$$

Gen. soln:

$$y = y_p + \underbrace{Ay_1 + By_2}_{\checkmark}$$

y_p ↓ ?

<u>Strategy</u>: Trial & error, guided by the form of $r(x)$

"<u>Method of undetermined coefficients</u>"

We'll illustrate the idea & pitfalls for

$$y'' + py' + qy = Ae^{ax}$$

Note: A & a are given!

Given the form of $r(x)$ (15)

try an exponential soln

Try: Ansatz:

$$y_p = C e^{\boxed{a}x}$$

same a as in $r(x)$

undetermined coefficient

into ODE:

$$y'' + py' + qy = Ae^{ax}$$

$$C\cancel{e^{ax}}(a^2 + pa + q) \overset{!}{=} A\cancel{e^{ax}}$$

$$C = \frac{A}{a^2 + pa + q}$$

so:

$$y_p = \frac{A}{a^2 + pa + q} e^{ax}$$

is a soln. of the inhomog. ODE