$$\underbrace{y' - xy}_{p(x)} = \underbrace{x}_{q(x)}$$

$$y(x) = \underbrace{-1}_{y_P} + \underbrace{C \exp\left(\tfrac{1}{2}x^2\right)}_{y_H}$$

gen. soln; arb. const. C.

Observation:

soln. has two parts

$$y(x) = y_P + y_H$$

y_P = a(ny) particular soln. of the full eqn.

y_H = general soln. of the homogeneous ODE, i.e. the ODE for $q(x) = 0$.

In our example:

- $y_P = -1$ Check: $y_P' = 0$

into ODE $y' - xy \overset{?}{=} x$

$$(-x)(-1) \overset{?}{=} x \quad ✓$$

- y_H:

$$y_H' - x y_H = 0$$

$$\frac{dy_H}{dx} - x y_H = 0$$

$$\int \frac{1}{y_H} dy_H = \int x\,dx$$

$$\ln y_H = \frac{1}{2}x^2 + A$$

$$y_H = \exp\left(\frac{1}{2}x^2 + A\right)$$

$$= \underbrace{\exp(A)}_{C'} \exp\left(\frac{1}{2}x^2\right)$$

$$y_H = C' \exp\left(\frac{1}{2}x^2\right)$$

as claimed!

In fact this structure of the solution is generic for linear ODEs.

2^{nd} order ODEs

$$\boxed{\begin{gathered} F(x, y, y', y'') = 0 \\ + \text{ 2 BCs or 2 ICs} \end{gathered}}$$

Linear 2^{nd} order ODEs

standard forms:

$$a(x) y'' + b(x) y' + c(x) y = d(x)$$

if $a(x) \neq 0$

$$y'' + p(x) y' + q(x) y = r(x)$$

Note: Vanishing highest derivative always means "trouble".

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Some theory for *linear* 2nd order ODEs

Existence and Uniqueness

Consider the *linear* second-order ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x), \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the functions $p(x)$, $q(x)$ and $r(x)$ are given.

Theorem

> If the functions $p(x)$, $q(x)$ and $r(x)$ are continuous functions of x in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in the entire interval I.

Notes:

- This is the promised extension of the statement for first-order problems. The extension to even higher-order linear ODEs should be obvious...
- If the functions $p(x)$, $q(x)$ and $r(x)$ are "well-behaved" (no jumps, singularities, etc.), the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

The homogeneous ODE & superposition of its solutions

If we set $r(x) = 0$ in the *inhomogeneous* ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x), \qquad \text{(I)}$$

we obtain the corresponding *homogeneous* ODE

$$y'' + p(x)\,y' + q(x)\,y = 0. \qquad \text{(H)}$$

A trivial (?) but useful observation

If $y_1(x)$ and $y_2(x)$ are two solutions of (H) then the linear combination

$$y_3(x) = A\,y_1(x) + B\,y_2(x)$$

is also a solution, for any values of the constants A and B.

Linear independence

To see why this is a useful observation, we need to define the concept of linear independence: Two nonzero functions $y_1(x)$ and $y_2(x)$ are linearly independent if

$$A\,y_1(x) + B\,y_2(x) = 0 \quad \forall x \qquad \Longleftrightarrow \qquad A \equiv B \equiv 0$$

(...just as in linear algebra...).

Examples:

- $y_1(x) = x$ and $y_2(x) = 3\,x^2$ are linearly independent.
- $y_1(x) = x$ and $y_2(x) = 3\,x$ are linearly dependent – they're just multiples of each other.

$$y'' + p(x)y' + q(x)y = 0 \quad (H)$$

$$y = A y_1(x) + B y_2(x)$$

into ODE:

$$\underbrace{A y_1'' + B y_2''}_{y''} + p(x)\underbrace{\left(A y_1' + B y_2'\right)}_{y'} +$$

$$+ q(x)\underbrace{\left(A y_1 + B y_2\right)}_{y} \overset{?}{=} 0$$

$$A\overbrace{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}^{0} +$$

$$B\underbrace{\left(y_2'' + p(x)y_2' + q(x)y_2\right)}_{0 \text{ because } y_2 \text{ solves } (H)} \overset{?}{=} 0$$

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$y_1 = x$ & $y_2 = 3x^2$

are lin. indep.:

Proof:

$$A \underbrace{x}_{y_1} + B \underbrace{3x^2}_{y_2} \overset{!}{=} 0 \quad \forall x$$

check for $x = 1$:

$$\boxed{A + 3B = 0}$$

check for $x = -1$

$$\boxed{-A + 3B = 0}$$

add: $6B = 0 \Rightarrow B = 0$

into 1st eqn: $\Rightarrow A = 0$.

y_1 & y_2 are indeed lin. indep.

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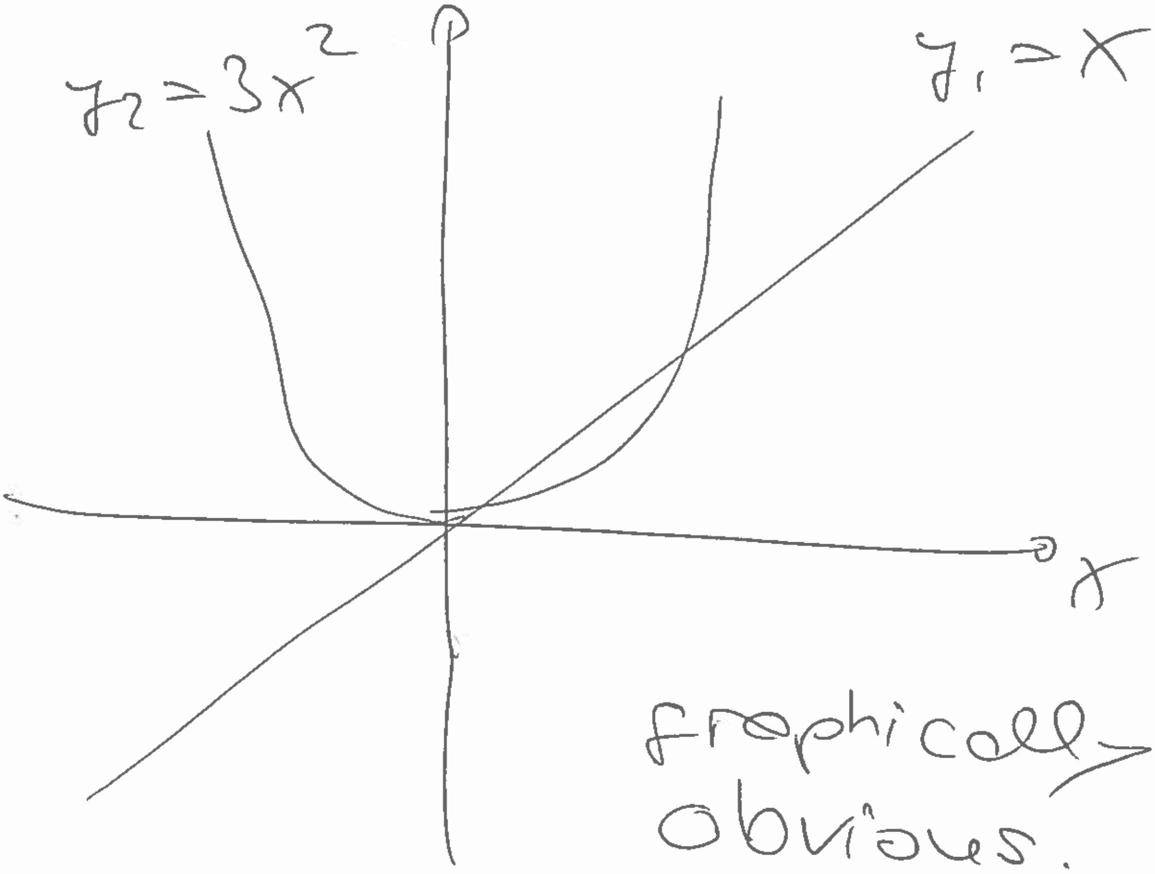


graphically obvious.

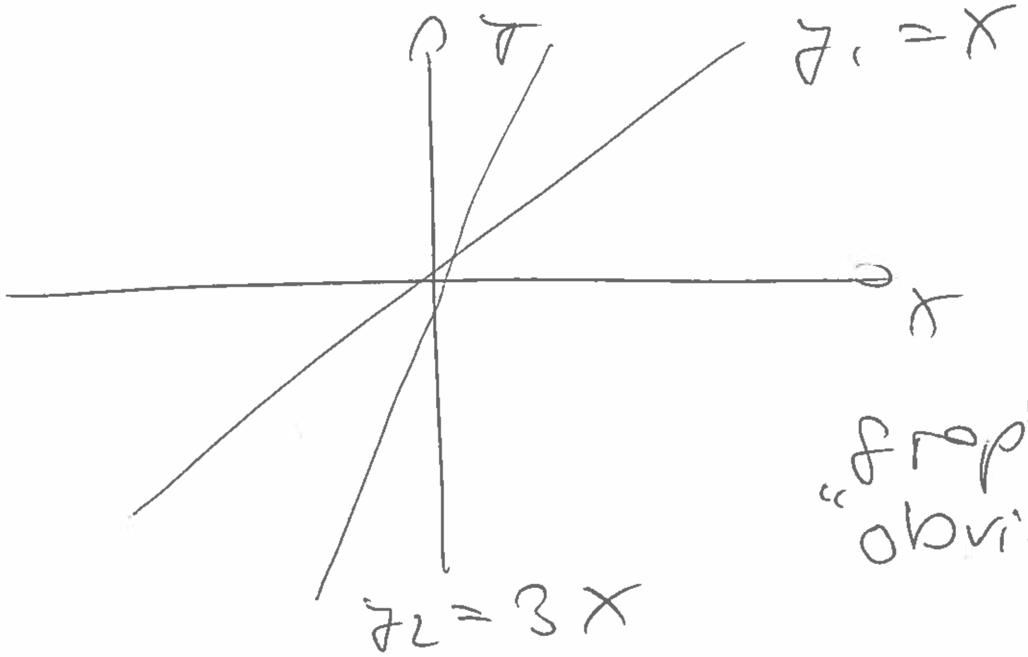
$y_1 = x \qquad y_2 = 3x$

$$A \underbrace{x}_{y_1} + B \underbrace{3x}_{y_2} = 0 \quad \forall x \; (*)$$

$$x(A + 3B) = 0$$

so choose e.g. $A = -3B \neq 0$ which shows that the cond. (*) can be satisfied for nonzero A & B.

$\Rightarrow$ y_1 & y_2 are lin. dep.



„graphically "obvious"

(fctns are multiples of each other)

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Fundamental solutions of the homogeneous ODE

Theorem

> *Any* solution of the homogeneous ODE
>
> $$y'' + p(x)\, y' + q(x)\, y = 0. \qquad \text{(H)}$$
>
> can be written as a linear combination of *any* two non-zero, linearly independent solutions, $y_1(x)$ and $y_2(x)$, say:
>
> $$y(x) = A\, y_1(x) + B\, y_2(x).$$

The two non-zero, linearly independent solutions $\{y_1(x),\, y_2(x)\}$ are called "fundamental solutions" of the homogeneous ODE (H).

Notes:

- The set of fundamental solutions is not unique!

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The general solution of the inhomogeneous ODE

Theorem

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + A\,y_1(x) + B\,y_2(x),$$

where:

- A and B are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Notes:

- Note the similarities between the structure of the solution of the linear ODE and the structure of the solution of the linear (algebraic) equation $\mathbf{Ax} = \mathbf{b}$. This is not accidental! There are deep connections between the two fields – matrices and the homogeneous part of a linear ODE are both "linear operators".
- The values of the constants A and B are determined by the boundary or initial conditions.

Example:

$$y'' + \underbrace{\frac{1}{x}}_{p(x)} y' - \underbrace{\frac{1}{x^2}}_{q(x)} y = \underbrace{-\frac{1}{x^2}}_{r(x)}$$

IC: $y(x=1) = 1$ $\qquad X = 1$

$y'(x=1) = 1$ $\qquad Y = 1$

$\qquad Z = 1$

$$\left.\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \\ r(x) &= -\frac{1}{x^2} \end{aligned}\right\}$$

these three fcts are continuous fcts of x in

$I_1 = (-\infty, 0)$

$I_2 = (0, \infty)$

$X \in I_2$

$\Rightarrow$ unique soln exists for $x > 0$.

- particular soln: y_p

$y_p = 1$ is a soln of the ODE.

Note: $y_p = 1 + x$ is another one

- ~~homog~~ solutions to the homog. ODE (= "homog. solns")

$y_1 = x$

$y_2 = \frac{1}{x}$

are both solutions of the homog. ODE

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$$

Check for

$y_1 = x$

$y_1' = 1$

$y_1'' = 0$

$$0 + \frac{1}{x} 1 - \frac{1}{x^2} x \overset{?}{=} 0 \quad \checkmark$$

$y_1(x) = x \qquad y_2(x) = \frac{1}{x}$

are nonzero solns of the homog ODE. Are they lin. indep.?

$$\underbrace{A\,x}_{y_1} + \frac{B}{x} \overset{!}{=} 0 \qquad \forall x$$

check at $x=1$: $\quad 2A + 2B = 0$

$x = 2 \quad 2A + \frac{1}{2}B = 0$

$$\frac{3}{2}B = 0$$

$$\Rightarrow B = 0$$

into (1): $A = 0$

$\Rightarrow y_1$ & y_2 are lin. indep.

$\Rightarrow \left\{ x, \frac{1}{x} \right\}$ are fundamental solutions of the homogeneous ODE.

So the general soln. of the ODE is

$$y = \underbrace{1}_{y_p} + A\underbrace{x}_{y_1} + B\underbrace{\frac{1}{x}}_{y_2}$$

Apply IC:

$$y' = A - Bx^{-2}$$

$y(x=1) = 1$: $1 + A + B = 1$

$y'(1) = 1$: $A - B = 1$

$$1 + 2A = 2$$

$$\Rightarrow A = \frac{1}{2}$$

$$\Rightarrow B = -\frac{1}{2}$$

$$\underline{\underline{y = 1 + \frac{1}{2}x - \frac{1}{2}\frac{1}{x}}}$$

is the unique soln.

Note: soln has singularity

at $x=0$.

<u>Q:</u> What would have happened if we had chosen different y_p, y_1, y_2?

<u>E.g.:</u> $y_p = 1+x$ solves ODE.

$$\left.\begin{aligned} y_1 &= x \\ y_2 &= x + \frac{2}{x} \end{aligned}\right\} \text{ solve the homog. ODE.}$$

y_1 & y_2 are nonzero & lin. indep.

So can also write the gen. soln as:

$$y = \underbrace{1+x}_{y_p} + C'\underbrace{x}_{y_1} + D\underbrace{\left(x+\frac{2}{x}\right)}_{y_2}$$

$$y' = 1 + C' + D\left(1 - 2x^{-2}\right)$$

Apply ICs:

$$y(1) = 1 = 1 + 1 + C + D(1+2)$$

$$y'(1) = 1 = 1 + C + D(1-2)$$

$$C + 3D = -1$$

$$C - D = 0$$

$$4D = -1 \Rightarrow D = -1/4$$

$$C = -1/4$$

unique soln is:

$$y = \underbrace{1 + x}_{y_p} - \frac{1}{4}\underbrace{x}_{y_1} - \frac{1}{4}\underbrace{\left(x + \frac{2}{x}\right)}_{y_2}$$

$$\underline{\underline{y = 1 + \frac{1}{2}x - \frac{1}{2}\frac{1}{x}}}$$ as before!

Note: soln does blow up at $x=0$, consistent with the E & U theorem which guaranteed a unique soln to exist in $I_1 = (0, +\infty)$

However, the solution may still exist for all values of $x \in \mathbb{R}$ if we use different ICs.

$$y = 1 + Ax + B\frac{1}{x}$$

may be nonsingular if $B=0$

E.g.: $y(1) = \frac{3}{2}$

$y'(1) = \frac{1}{2}$

$$\Rightarrow \quad y = 1 + \frac{1}{2}x$$

which is continuous & exists for all $x \in \mathbb{R}$.