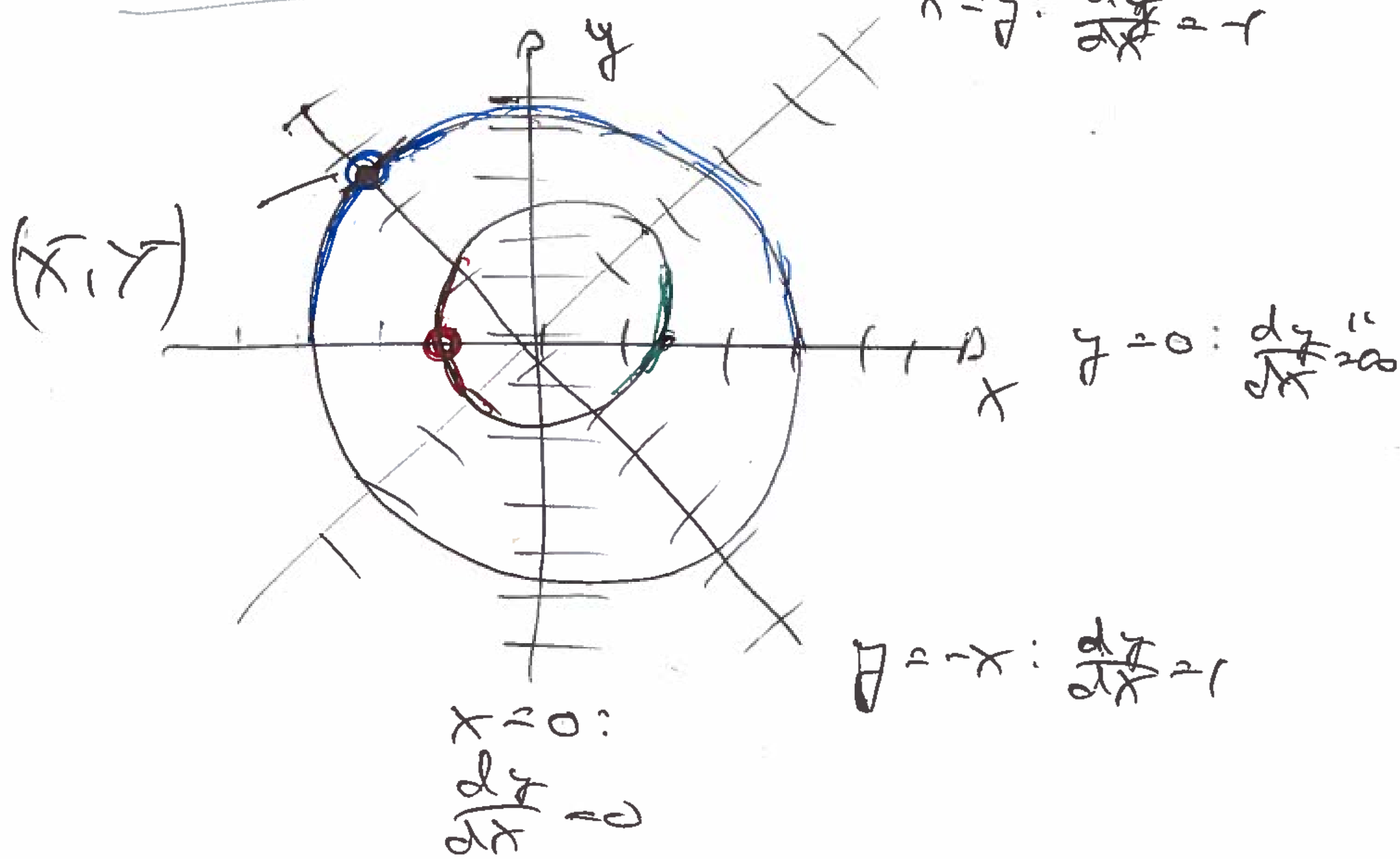
1

$$\boxed{y' = -\frac{x}{y}} = f(x,y)$$



$$y = \pm\sqrt{R^2 - x^2}$$ ✓

$$\left.\begin{aligned} f(x,y) &= -x y^{-1} \\ \frac{\partial f}{\partial y} &= +x y^{-2} \end{aligned}\right\}$$ continuous fcts of (x,y) in the vicinity of (x,y) if $y \neq 0$

If $y \neq 0$: soln does indeed exist in the vicinity of (x, y) – see sketch – but only for a finite range of x-values.

If $y = 0$: soln. can only be extended from I.C. to left or right. Also: we can pick the upper or lower branch $\rightarrow$ non-uniqueness.

$(x, y) = (0, 0)$ is a "critical point", a point where multiple isoclines intersect.

Separable ODEs

1^{st} order ODE is separable if it can be re-arranged into the form:

$$g(y)\frac{dy}{dx} = h(x)$$

$$\int g(y(x))\frac{dy}{dx}\,dx = \int h(x)\,dx$$

chain rule

$$\int g(y)\,dy = \int h(x)\,dx + A$$

arbitrary constant from I.C.

Note: Not always possible to solve explicitly for $y(x)$.

Example:

$$\frac{dy}{dx} = y' = -\frac{x}{y} \qquad \text{(revisited)}$$

$$y(x=1) = 0$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{1}{2}y^2 + C = -\frac{1}{2}x^2 + \hat{A}$$

$$\frac{1}{2}(x^2+y^2) = \underbrace{\hat{A} - C}_{A}$$

$$x^2 + y^2 = R^2$$

arb. constant

$$y(x) = \pm\sqrt{R^2 - x^2}$$

IC: $y(x=1) = 0 \Rightarrow R = 1$

$$\Rightarrow y(x) = \pm\sqrt{1-x^2}$$

not unique!

If we had imposed

$$y(x=0) = 1:$$

$$y(x) = +\sqrt{1-x^2}$$

unique.

<u>Example</u>: $$\frac{dy}{dx} = \exp(x-y)$$

$$y(x_0) = y_0$$

$$\frac{dy}{dx} = \exp(x)\exp(-y)$$

$$\int e^y\,dy = \int e^x\,dx$$

$$e^y = e^x + A$$

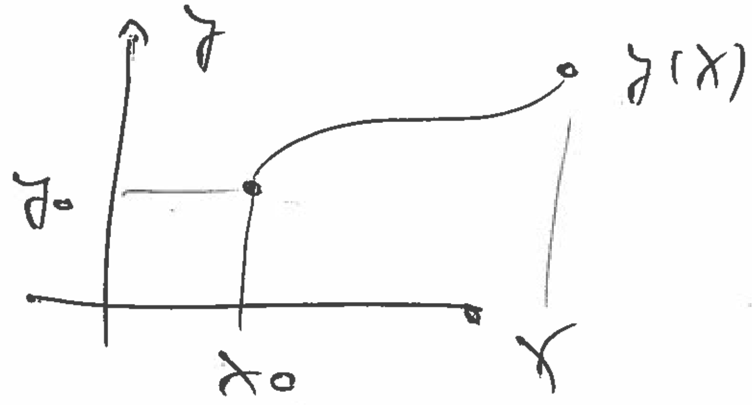
(6)

$$y(x) = \ln(e^x + A)$$

Apply IC to determine A.

Alternative: Definite integrals:

$$\int e^y \, dy = \int e^x \, dx$$



$$\int_{y_0}^{y(x)} e^{\eta} \, d\eta = \int_{x_0}^{x} e^{\xi} \, d\xi$$

$$\exp(y(x)) - \exp(y_0) = \exp(x) - \exp(x_0)$$

$$y(x) = \ln\left(\exp(x) - \exp(x_0) + \exp(y_0)\right)$$

Example:

$$(1-y^2)\sin x\frac{dy}{dx} - y\cos x = 0$$

$$\frac{1-y^2}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x}$$

$$\int\left(\frac{1}{y} - y\right)dy = \int \cot x\,dx + A$$

$$\ln|y| - \frac{1}{2}y^2 = \ln|\sin x| + A$$

cannot be solved explicitly for $y(x)$ but here we can find the inverse of the form:

$$x = \arcsin\left(B|y|e^{-\frac{y^2}{2}}\right)$$

(EXERCISE)

1st order ODEs of homogeneous type

ODE is of homog. type if it can be re-arranged into the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Trick: Substitute:

$$\frac{y(x)}{x} = z(x)$$

$$y(x) = x\, z(x)$$

$$\frac{dy}{dx} = \underbrace{x\,\frac{dz}{dx} + z = f(z)}$$

$$x\,\frac{dz}{dx} + z = f(z)$$

ODE for $z(x)$ which can be solved by separation

$$\frac{dz}{dx} = \frac{f(z) - z}{x}$$

$$\int \frac{dz}{f(z)-z} = \int \frac{1}{x}\,dx + A$$

DO <u>NOT</u> MEMORISE THIS!

<u>Example:</u>

$$x^2 \frac{dy}{dx} + y^2 - xy = 0 \quad \Big|\cdot \frac{1}{x^2}$$

$$\frac{dy}{dx} + \left(\frac{y}{x}\right)^2 - \frac{y}{x} = 0$$

<u>Note:</u> $x \neq 0$

(<u>Note:</u> $y=0$ is a soln).

ODE is of homog. type

$$z = \frac{y}{x} = z(x) \; ; \; y(x) = x \cdot z(x)$$

$$\frac{dy}{dx} = x\frac{dz}{dx} + z$$

into ODE:

$$\underbrace{x\frac{dz}{dx}+\cancel{z}}_{\frac{dy}{dx}} + \underbrace{z^2}_{\left(\frac{y}{x}\right)^2} - \underbrace{\cancel{z}}_{\frac{y}{x}} = 0$$

$$x\frac{dz}{dx} = -z^2$$

$$\int -\frac{1}{z^2}\,dz = \int \frac{1}{x}\,dx$$

$$\frac{1}{z} = \ln|x| + C = \frac{x}{y(x)}$$

$$y(x) = \frac{x}{\ln|x| + C}$$

Linear 1st order ODEs

Standard form

$$y' + p(x)\,y = q(x)$$

can be solved by an integrating factor.

<u>Idea:</u> multiply ODE by some fct $\mu(x)$ & integrate.

$$\int \underbrace{(y'\mu + y\,p\mu)}\,dx = \int q\mu\,dx$$

<u>if</u> this was:

$$y'\mu + y\mu' = \frac{d}{dx}(\mu y)$$

$$\int \frac{d}{dx}(\mu y)\,dx = \mu y = \int q\mu\,dx$$

$$y(x) = \frac{1}{\mu(x)}\int q(x)\mu(x)\,dx$$

So to make this work choose $\mu(x)$ so that

$$\frac{d\mu}{dx} = p(x)\mu(x)$$

This a separable ODE for $\mu(x)$.

$$\int \frac{1}{\mu} d\mu = \int p \, dx$$

$$\ln \mu = \int p(x) \, dx$$

$$\mu(x) = \exp\left(\int p(x) \, dx \right)$$

can ignore the const. of integration.

Procedure:

$$y' + p(x) y = q(x)$$

1. Determine integrating factor:

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

can ignore const. of integration

2. multiply ODE by $\mu(x)$ & integrate

$$\underbrace{\int \left(y' + p(x)y\right)\mu(x)\,dx}_{\mu(x)\,y(x)} = \int q(x)\mu(x)\,dx + C$$

3. $$y(x) = \frac{\int q(x)\mu(x)\,dx + C}{\mu(x)}$$

const. of integration matters.

Exemplo:

$$y' - xy = x$$

$$p(x) = -x \qquad q(x) = x$$

① Int. factor.

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

$$= \exp\left(\int -x\,dx\right)$$

$$\mu(x) = \exp\left(-\tfrac{1}{2}x^2\right)$$

②

$y \quad \mu'$

$$\int \left[y' \exp\left(-\tfrac{1}{2}x^2\right) - x y \exp\left(-\tfrac{1}{2}x^2\right) \right] dx$$

$$= \int x \exp\left(-\tfrac{1}{2}x^2\right) dx$$

$$\int \underbrace{y'\mu + y\mu'}\,dx$$

$$= \int \frac{d(\mu y)}{dx}\,dx = \mu(x)\,y(x)$$

$$\exp\left(-\tfrac{1}{2}x^2\right)y(x) = \underbrace{\int x \exp\left(-\tfrac{1}{2}x^2\right)dx}_{-\exp\left(-\frac{x^2}{2}\right)+C}$$

$$\underline{\underline{y(x) = -1 + C\exp\left(\tfrac{1}{2}x^2\right)}}$$

general soln; const. C determined from IC.