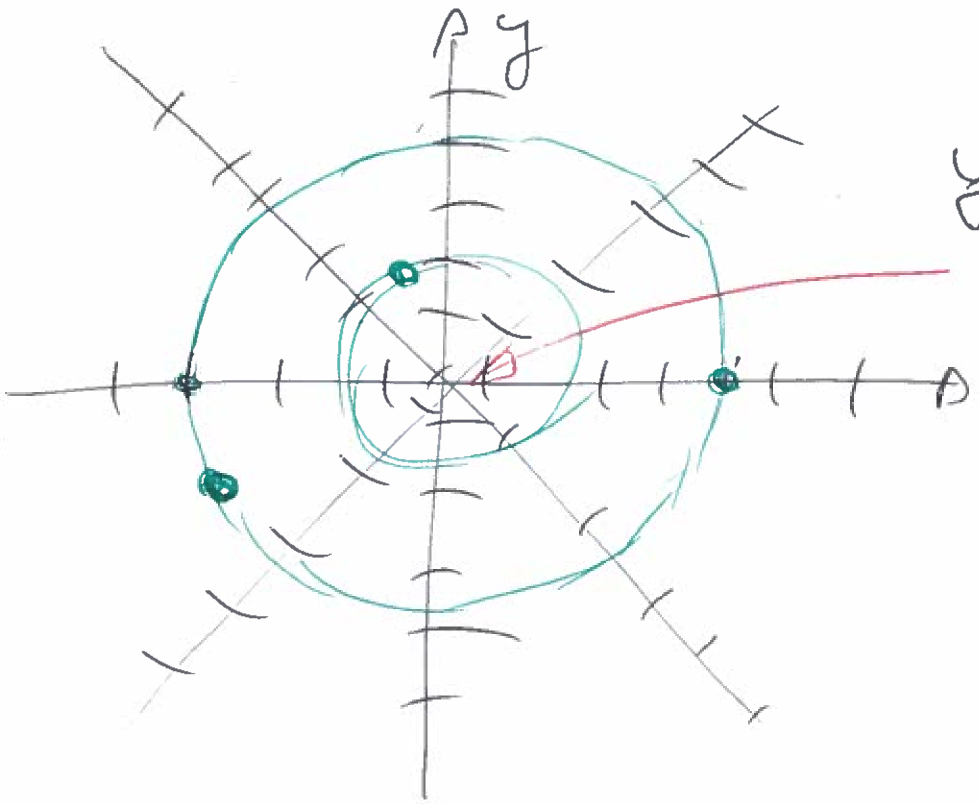
$y' = -\frac{x}{y} = f(x,y)$ ← slope of soln. curves



- solns: circular arcs
- E & U: locally unless $\bar{y} = 0$

Recall theory:

$$\left.\begin{aligned} f(x,y) &= -xy^{-1} \\ \frac{\partial f}{\partial y} &= xy^{-2} \end{aligned}\right\}$$ cont. fcts of x & y unless $y = 0$

Separable ODE

1^{st} order ODE is separable if it can be re-arranged into the form:

$$g(y)\frac{dy}{dx} = h(x)$$

$$\int g(y)\frac{dy}{dx}\,dx = \int h(x)\,dx$$

$$\int g(y(x))\frac{dy}{dx}\,dx$$

Change of variables

$$\int g(y)\,dy = \int h(x)\,dx + A$$

for an arbitrary constant A.

<u>Note:</u> Not always possible to "do" the integrals or solve for $y(x)$ explicitly.

$y' = -\frac{x}{y}$ $\quad y(x=1) = 0$

$\frac{dy}{dx} = -\frac{x}{y}$

$\int y\,dy = -\int x\,dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + A$

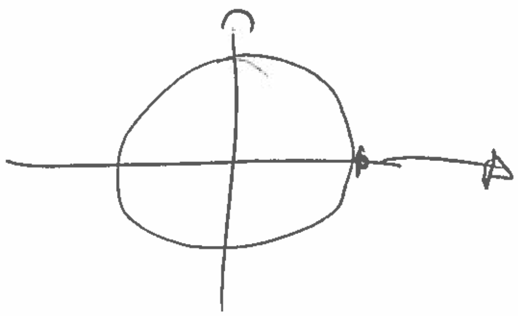
$2A = R^2$

$x^2 + y^2 = R^2$

$\cancel{y^2}$ $\quad y = \pm\sqrt{R^2 - x^2}$

Apply IC: $y(x=1) \stackrel{!}{=} 0 \Rightarrow R = 1$

$y = \pm\sqrt{1 - x^2}$



Ex:
$$\frac{dy}{dx} = e^{x-y} \qquad y(x_0) = y_0$$

$$= e^{x} e^{-y}$$

$$\int e^{y}\,dy = \int e^{x}\,dx$$

$$e^{y} = e^{x} + A$$

$$y(x) = \ln(e^{x} + A)$$

Now apply IC to determine A.

Alternative: Definite integral

$$\int_{y_0}^{y(x)} e^{\eta}\,d\eta = \int_{x_0}^{x} e^{\xi}\,d\xi$$

$$e^{y(x)} - e^{y_0} = e^{x} - e^{x_0}$$

$$e^{y(x)} = e^{x} - e^{x_0} + e^{y_0}$$

$$y(x) = \ln\left(e^{x} - e^{x_0} + e^{y_0}\right)$$

Ex:

$$(1-y^2)\sin x \frac{dy}{dx} - y\cos x = 0$$

$$\int \frac{1-y^2}{y}\, dy = \int \frac{\cos x}{\sin x}\, dx$$

$$\int \left(\frac{1}{y} - y\right) dy = \int \cot x \, dx + A$$

$$\ln|y| - \frac{1}{2}y^2 = \ln|\sin x| + A$$

cannot be solved explicitly for $y(x)$, but (EXERCISE)

$$x = \arcsin\left(B\,|y| \exp\left(-\frac{1}{2}y^2\right)\right).$$

1^{st} order ODEs of homogeneous type

ODE is of homogeneous type if it can be written as:

$$\frac{dy}{dx} = f\Big(\underbrace{\frac{y}{x}}_{z}\Big)$$

Trick:

Def: $z = \frac{y}{x}$; $z(x) = \frac{y(x)}{x}$

$y(x) = x\, z(x)$

$\frac{dy}{dx} = z + x\frac{dz}{dx}$ into ODE

$$z + x\frac{dz}{dx} = f(z)$$

1^{st} order separable ODE for $z(x)$:

$$\frac{dz}{dx} = \frac{f(z) - z}{x}$$

$$\int \frac{dz}{f(z)-z} = \int \frac{dx}{x}$$

No point in remembering this!

Ex:

$$x^2 \frac{dy}{dx} + y^2 - xy = 0 \quad \Big| \cdot \frac{1}{x^2}$$

Note: $y=0$ is a soln.

Note: $x \neq 0$

$$\frac{dy}{dx} + \underbrace{\left(\frac{y}{x}\right)^2}_{z^2} - \underbrace{\frac{y}{x}}_{z} = 0$$

$$z(x) = \frac{y(x)}{x} \qquad y(x) = x\, z(x)$$

$$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

into ODE:

$$\underbrace{\cancel{z} + x\frac{dz}{dx}}_{\frac{dy}{dx}} + z^2 - \cancel{z} = 0$$

$$x\frac{dz}{dx} = -z^2$$

$$\int -\frac{1}{z^2}\,dz = \int \frac{1}{x}\,dx$$

$$\frac{1}{z(x)} = \ln|x| + C$$

$$z(x) = \frac{y(x)}{x}$$

$$\frac{1}{y} = \frac{\ln|x| + C}{x}$$

$$y(x) = \frac{x}{\ln|x| + C}$$

Note: $y = 0$ (the soln. we had already spotted) is not a special case of this soln

Linear 1st order ODEs

Standard form

$$y' + p(x) y = q(x)$$

Can be solved by an integrating factor:

<u>Motivation:</u>

multiply ODE by some fct $\mu(x)$ & integrate:

$$\int \underbrace{(y'\mu + y\, p\mu)}\, dx = \int \mu q \, dx$$

If $p\mu = \mu'$ (*) then

~~$y'\mu + \mu y'$~~ $\int (y'\mu + y\mu')\, dx$

$$\int \frac{d}{dx}(y\mu)\, dx = y\mu$$

To make (*) true:

$$\mu' = \frac{d\mu}{dx} = p(x)\,\mu(x)$$

$$\int \frac{d\mu}{\mu} = \int p(x)\,dx$$

$$\ln|\mu| = \int p(x)\,dx$$

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

No need for constant of integration!

Procedure:

(1) Determine integrating factor:

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

(2) Multiply ODE by this $\mu(x)$ & integrate both sides.

$$\underbrace{\int (y' + p y)\mu \, dx}_{\mu(x)\, y(x)} = \int q \mu \, dx \quad (11)$$

$$\boxed{y(x) = \frac{1}{\mu(x)} \left\{ \int q(x)\mu(x)\, dx + C \right\}}$$

<u>Ex:</u>

$$y' \underbrace{- x}_{p(x)} y = \underbrace{x}_{q(x)}$$

1. integrating factor:

$$\mu(x) = \exp\left(\int p(x)\, dx \right)$$

$$= \exp\left(\int -x\, dx \right) = \exp\left(-\frac{1}{2} x^2 \right)$$

2.

$$\int \left[y' \exp\left(-\tfrac{1}{2}x^2\right) + y\left(-x \exp\left(-\tfrac{1}{2}x^2\right)\right)\right] dx =$$

$$= \int x \exp\left(-\tfrac{1}{2}x^2\right) dx + C$$

$$\int \frac{d}{dx}\left(y \exp\left(-\tfrac{1}{2}x^2\right)\right) dx = \int x \exp\left(-\tfrac{1}{2}x^2\right) dx + C$$

$$y \underbrace{\exp\left(-\tfrac{1}{2}x^2\right)}_{\mu(x)}$$

$$\int x \exp\left(-\tfrac{1}{2}x^2\right) dx = -\exp\left(-\frac{x^2}{2}\right) + C$$

$$y(x) \exp\left(-\tfrac{1}{2}x^2\right) = -\exp\left(-\tfrac{1}{2}x^2\right) + C$$

$$\underline{\underline{y(x) = -1 + C \exp\left(\tfrac{1}{2}x^2\right)}}$$