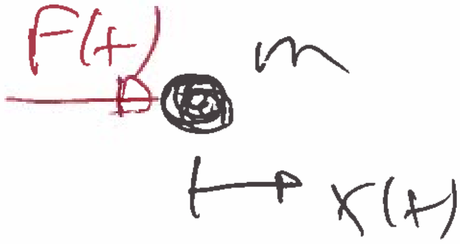


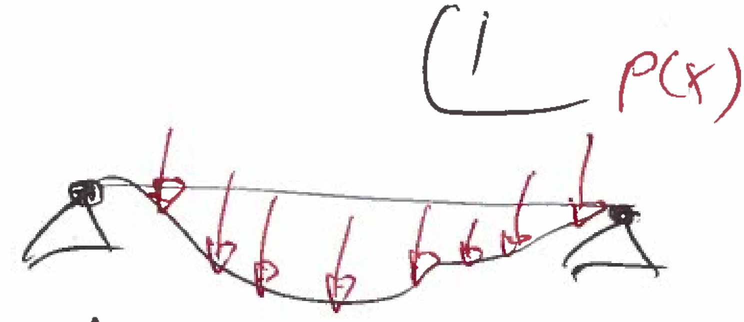
$$m\frac{d^2x}{dt^2}=f(t)$$

IC: $x(t=0)=x_0$

$$\left.\frac{dx}{dt}\right|_{t=0}=v_0$$

If $f(t)=f_0$:

$$x(t)=\frac{1}{2}\frac{f_0}{m}t^2+v_0t+x_0$$



y, x $\quad y=h(x)$

$$T\frac{d^2h}{dx^2}=p(x)$$

BC: $h(x=0)=0$

$h(x=L)=0$

If $p(x)=p_0$

$$h(x)=\frac{1}{2}\frac{p_0}{T}(x^2-Lx)$$

E & U "obvious"

IVP

BVP

Finally, a counterexample:

$$y' = y^{1/2} \qquad y(0) = 0$$

One IC for a 1st order ODE

 ?

Spot 2 solns:

$$y = 0$$

and

$$y = \frac{1}{4}x^2$$

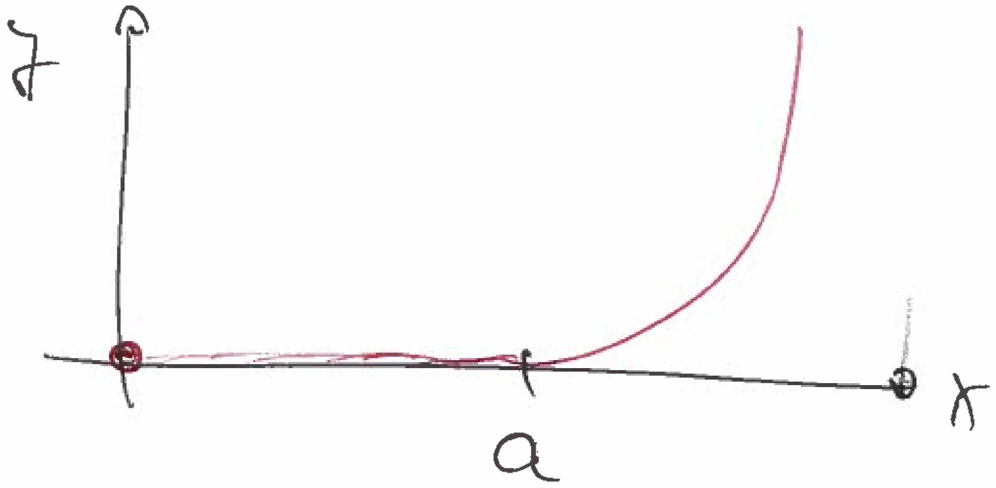
$$y' = \frac{1}{2}x \; ; \quad y^{1/2} = \frac{1}{2}x$$

$$\left.\begin{array}{l} y' \overset{?}{=} y^{1/2} \\ \text{IC} \checkmark \end{array}\right\} \checkmark$$

In fact:

$$y = \begin{cases} 0 & \text{for } 0 \le x \le a \\ \frac{1}{4}(x-a)^2 & \text{for } x > a \end{cases}$$

13



∞ many solns!

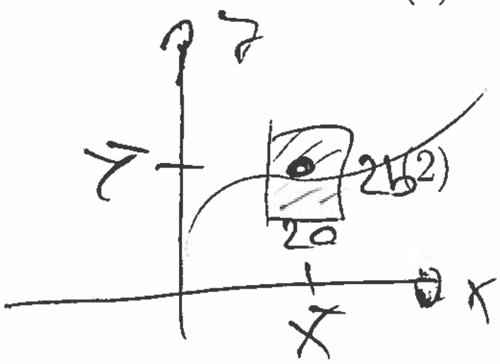
Existence and uniqueness theorem for 1st order ODEs

Consider the first-order ODE in its explicit form

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

subject to the initial condition

$$y(X) = Y, \tag{2}$$



where the constants X and Y are given.

Theorem

> If $f(x, y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous functions of x and y in a region $0 < |x - X| < a$ and $0 < |y - Y| < b$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

Notes:

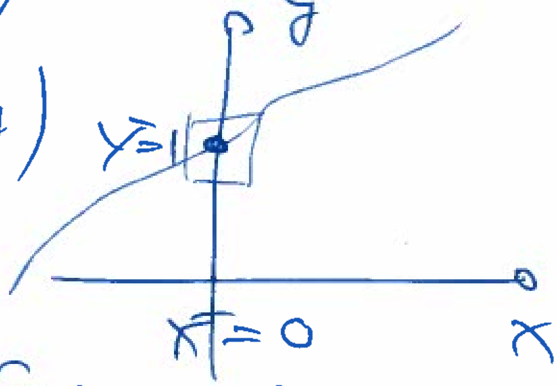
- The statement is easily generalised to higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

A pretty weak statement then....

Examples:

① $y' = \underbrace{\sin(xy)}_{f(x,y)}$; $y(0) = 1$

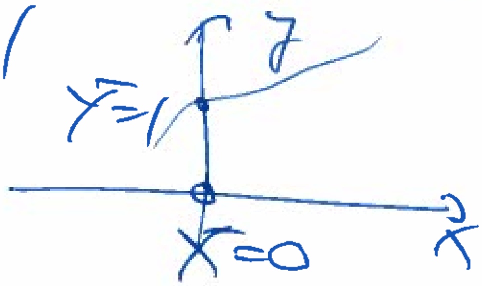
$$\begin{cases} f(x,y) = \sin(xy) \\ \dfrac{\partial f}{\partial y} = x\cos(xy) \end{cases}$$



both continuous fcts of x & y for all x & y

$\Rightarrow$ unique soln. exists in the vicinity of (X, Y).

② $y' = \underbrace{y^2}_{f(x,y)}$ $\quad y(0) = 1$



$$\left.\begin{array}{l} f(x,y) = y^2 \\ \dfrac{\partial f}{\partial y} = 2y \end{array}\right\}$$ cont. fcts for all values of x & y

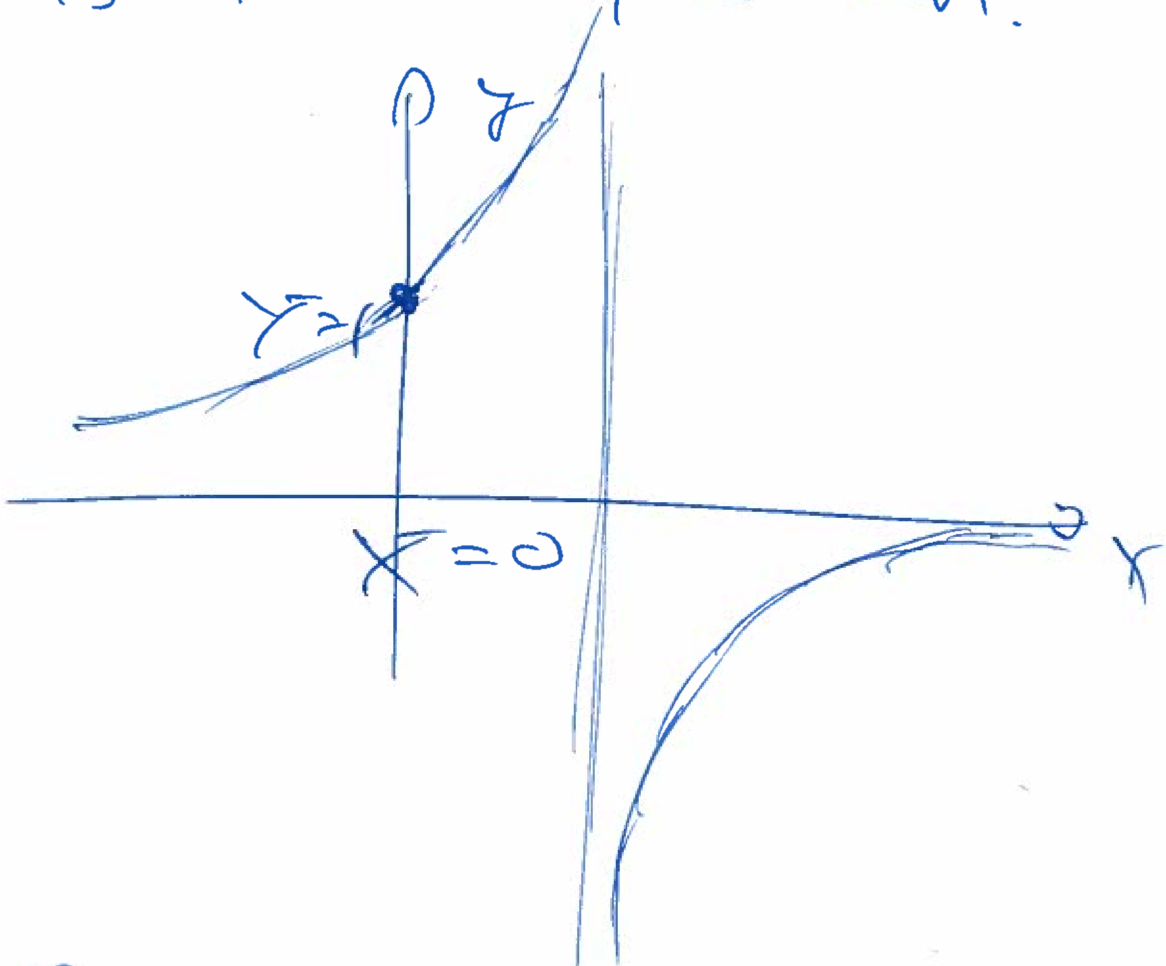
$\Rightarrow$ unique soln exists in

vicinity of $(0,1)$.

In fact,

$$y(x) = \frac{1}{1-x} = (1-x)^{-1}$$

is that unique soln.



Soln only exists for $x < 1$

Existence and uniqueness theorem for *linear* 1st order ODEs

Consider the *linear* first-order ODE

$$\frac{dy}{dx} + p(x)\, y = q(x), \tag{3}$$

subject to the initial condition

$$y(X) = Y, \tag{4}$$

where the constants X and Y and the functions $p(x)$ and $q(x)$ are given.

Theorem

> If the functions $p(x)$ and $q(x)$ are continuous functions in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (3) and (4) in the entire interval I.

Notes:

- The statement is again easily generalised to higher-order ODEs.
- The theorem provides a "much more global" statement. In fact, if the functions $p(x)$ and $q(x)$ are "well-behaved" (no jumps, singularities, etc.) the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

This is a much stronger statement and explains in part why (some) mathematicians love (only) linear problems.

Examples:

① $\underbrace{u' + x}_{p(x)} u = \underbrace{x}_{q(x)}$; $u(0) = 2$

$\left.\begin{array}{l} p(x) = x \\ q(x) = x \end{array}\right\}$ both are continuous fcts of x for $x \in \mathbb{R}$

$\Rightarrow$ unique soln. exists for $x \in \mathbb{R}$

$$\boxed{u(x) = 1 + \exp\left(-\frac{1}{2}x^2\right)}$$

② $u' + \underbrace{\frac{1}{x}}_{p(x)} u = \underbrace{2}_{q(x)}$ + I.C.

$\left.\begin{array}{l} p(x) = \frac{1}{x} \\ q(x) = 2 \end{array}\right\}$ continuous in two distinct intervals $I_1 = (-\infty, 0)$, $I_2 = (0, \infty)$

$\Rightarrow$ unique soln exists in either of these intervals!

In fact:

$$u(x) = x + \frac{A}{x}$$

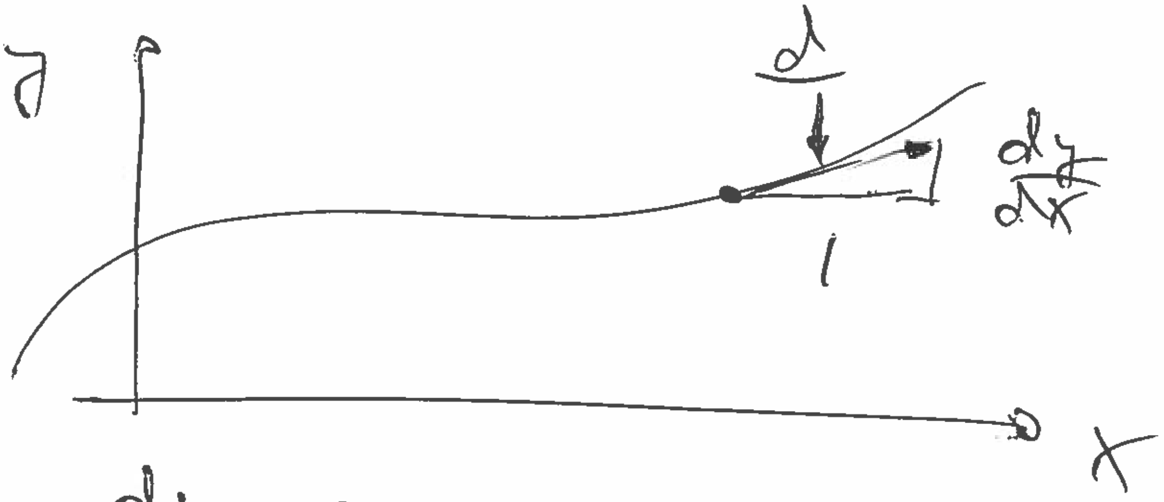
is the (most) general soln. It contains one constant, A, which is determined by the IC.

<u>Note:</u> In general the soln. does indeed not exist for $x = 0$.

First order ODEs

$$y' = f(x, y)$$

(I) Graphical approach



$\frac{dy}{dx}$ is the slope of the soln.

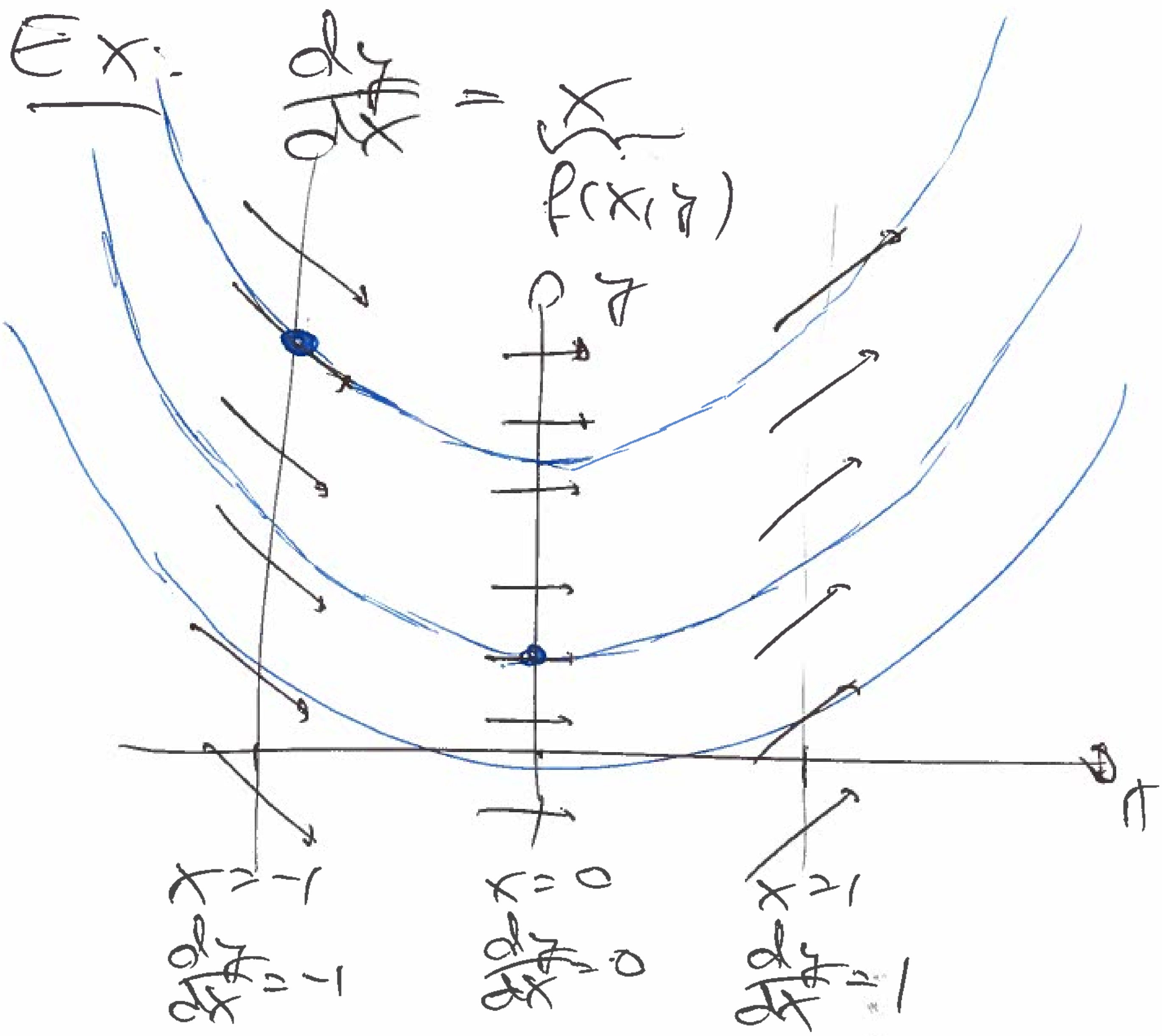
$f(x, y)$ defines the slope of solution curve(s).

<u>Def:</u> The "<u>direction field</u>" of the ODE is the set of all vectors that have the same direction as

$$\underline{d} = \begin{pmatrix} 1 \\ \frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} 1 \\ f(x, y) \end{pmatrix}$$

Def: "Integral curves" are curves that are everywhere tangent to the direction field. These integral curves are solns. of the ODE.

Ex: $\frac{dy}{dx} = \underbrace{x}_{f(x,y)}$



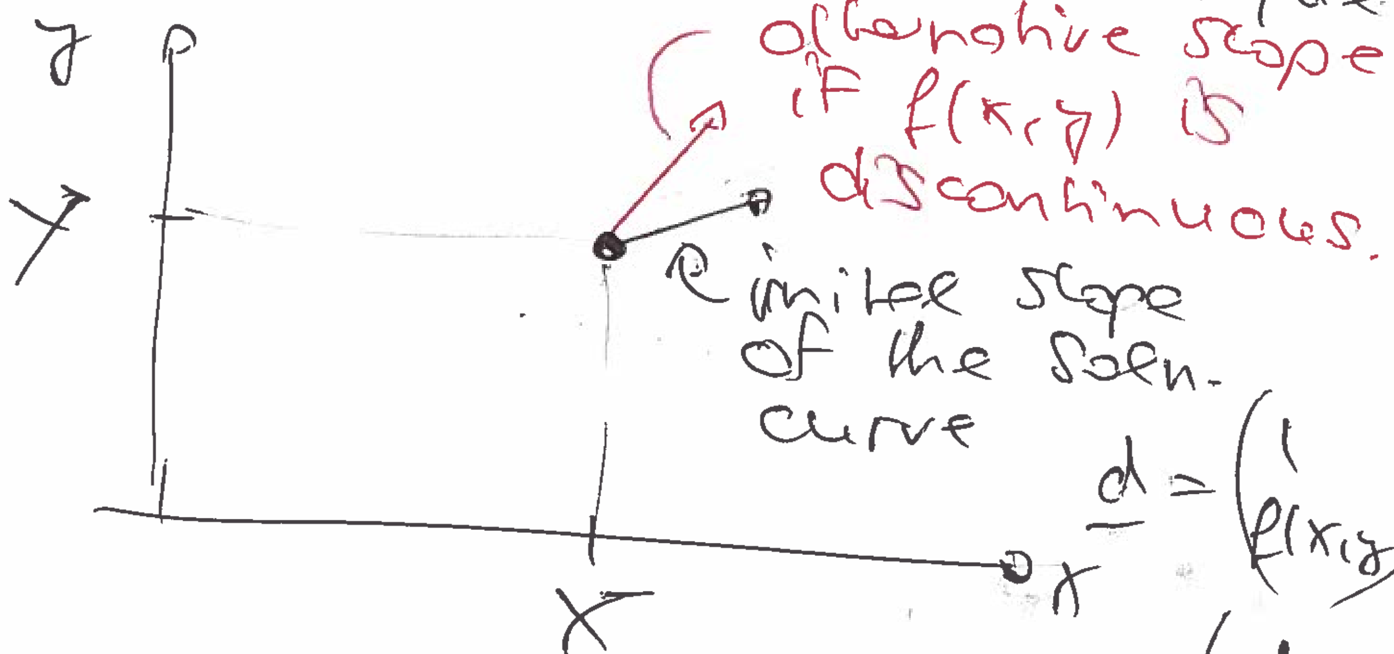
E & U is obvious & graphical soln illustrates the character of the soln.

In fact:

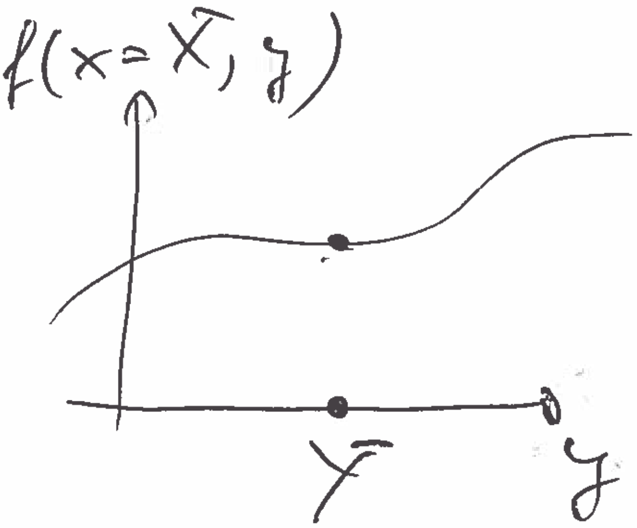
$$y = \frac{1}{2}x^2 + C'$$

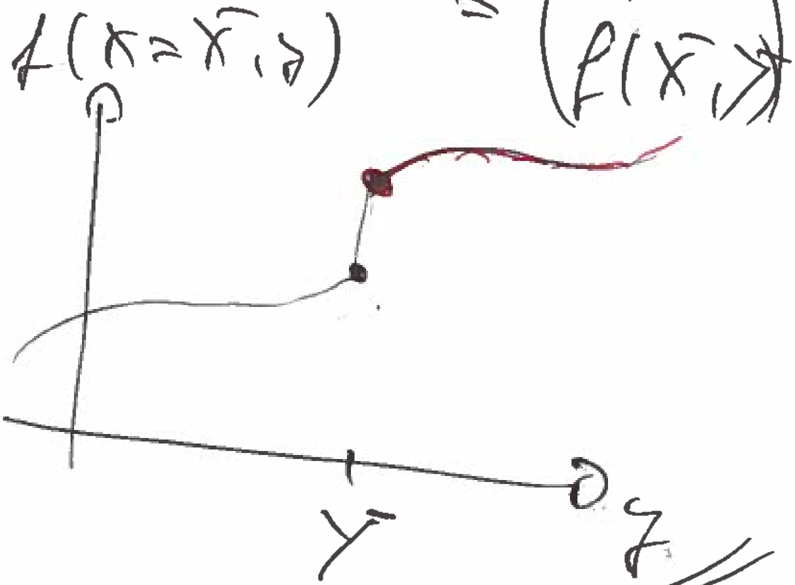
is the soln.

In fact: How can a solution starting from a given I.C. not be unique?



$$d = \begin{pmatrix} 1 \\ f(x,y) \end{pmatrix} = \begin{pmatrix} 1 \\ f(X,Y) \end{pmatrix}$$





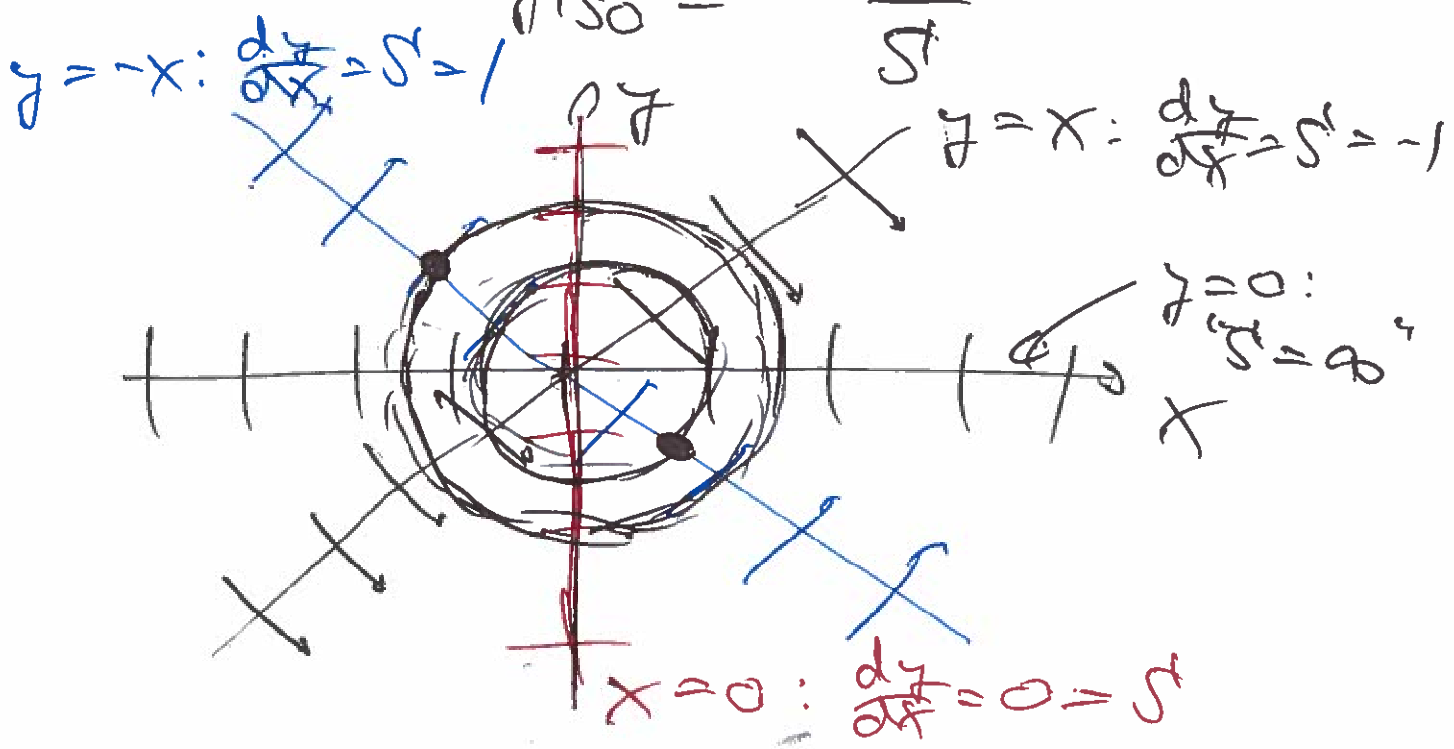
Observation:

When sketching solns it is helpful to identify so-called <u>iso-clines</u> which are lines in x-y plane where $\frac{dy}{dx} = \text{const.}$

Example:

$$\frac{dy}{dx} = \boxed{-\frac{x}{y}} = f(x,y) = \underbrace{s'}_{\text{const.}}$$

Isolines: $y_{iso} = -\frac{x}{s'}$



Solution: circular arcs (?)

In fact,

$$y = \pm\sqrt{R^2 - x^2}$$

is the general soln.

Const. R is determined by the IC.