Existence and Uniqueness for *non-linear* 2nd order ODEs

Consider the *non-linear* second-order ODE

$$y'' = f(x, y, y') \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the function $f(x, y, y')$, are given.

Theorem

> If $f(x, y, y')$ and $\frac{\partial f(x,y,y')}{\partial y}$ and $\frac{\partial f(x,y,y')}{\partial y'}$ are continuous functions of x, y and y' in a region $0 < |x - X| < a$, $0 < |y - Y| < b$ and $0 < |y' - Z| < c$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

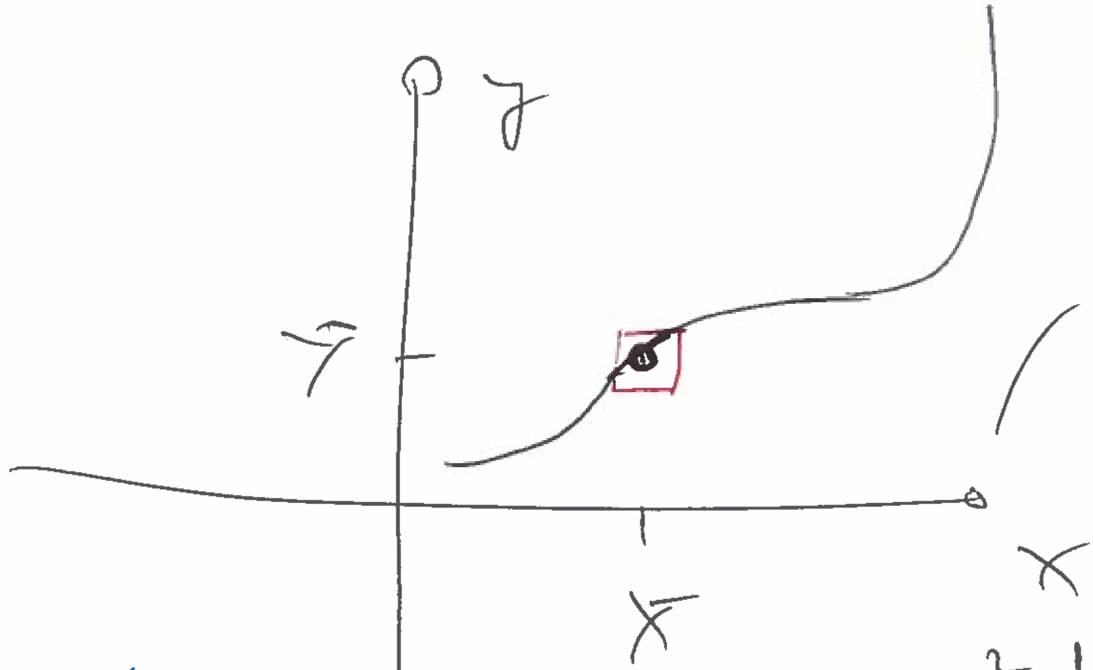
Notes:

- The statement is easily generalised to (even) higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

"Bootstrapping"

The theorem only guarantees the existence and uniqueness in the "vicinity" of the initial condition. However, if you can show that the function $f(x, y, y')$ and its derivatives are "well behaved" (in the sense of the theorem), for *any* values of x, y and y', then the repeated application of the theorem guarantees the existence and uniqueness of the solution for *all* values of x.

$$y' = f(x, y)$$



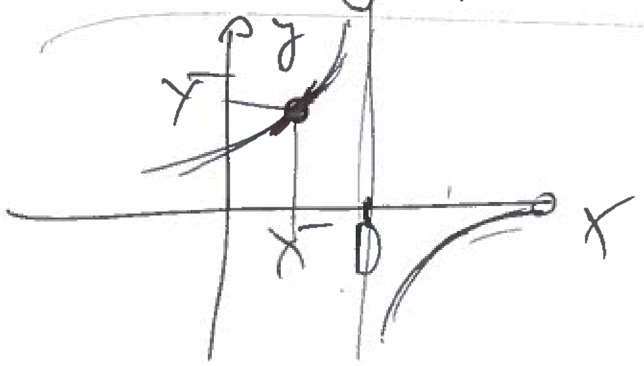
BUT IT'S SUBTLE:

$$\frac{dy}{dx} = \underbrace{y^2}_{f(x,y)}$$

$$\int y^{-2}\,dy = \int dx$$

$$-y^{-1} = x + C'$$

$$y(x) = \frac{1}{D - x}$$



$$y^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \underbrace{\frac{1}{y^2}}_{f(x,y)}$$

$$\int y^2\,dy = \int dx$$

$$\frac{1}{3} y^3 = x + C'$$

$$y = \sqrt[3]{3(x + C')}$$