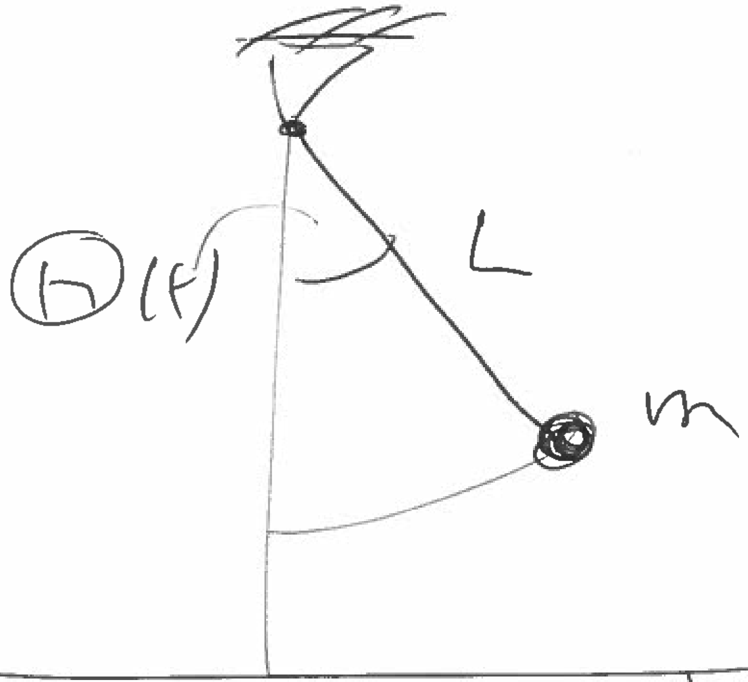


$$\ddot{\Theta} + \omega^2 \sin\Theta = 0$$

$$\omega^2 = \frac{g}{L}$$

IC: $\Theta(t=0) = \varepsilon$

$\dot{\Theta}(t=0) = 0$

Cannot be done!

E&U:

$$\ddot{\Theta} = \underbrace{-\omega^2 \sin\Theta}_{f(t,\Theta,\dot{\Theta})}$$

f, $\frac{\partial f}{\partial \Theta}$ & $\frac{\partial f}{\partial \Theta}$ are continuous fcts of their arguments

Existence and Uniqueness for *non-linear* 2nd order ODEs

Consider the *non-linear* second-order ODE

$$y'' = f(x, y, y') \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the function $f(x, y, y')$, are given.

Theorem

If $f(x, y, y')$ and $\frac{\partial f(x,y,y')}{\partial y}$ and $\frac{\partial f(x,y,y')}{\partial y'}$ are continuous functions of x, y and y' in a region $0 < |x - X| < a$, $0 < |y - Y| < b$ and $0 < |y' - Z| < c$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

Notes:

- The statement is easily generalised to (even) higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

$\Rightarrow$ unique soln exists at least for small values of t.

Form of the perturbation solution: from physics:

If $\Theta(t=0) = \varepsilon$, then

$|\Theta| \leq \varepsilon \ll 1$

$$\sin\Theta = \Theta - \cancel{\frac{1}{3!}\Theta^3} + \dots$$

In this limit the eqn becomes:

$$\boxed{\begin{array}{l} \ddot{\Theta} + \omega^2\Theta = 0 \\ \Theta(t=0) = \varepsilon; \ \dot{\Theta}(t=0) = 0 \end{array}}$$

$$\Theta(t) = \varepsilon\cos(\omega t)$$

This suggests posing an expansion as:

$$\boxed{\Theta(t) = \varepsilon\Theta_1(t) + \varepsilon^2\Theta_2(t) + \dots}$$

3

[EXERCISE: include $\Theta_0(t)$ into ansatz & show that $\Theta_0 \equiv 0$.]

Insert ansatz into ODE

$$\ddot{\Theta} + \underbrace{\omega^2}_{=1 \text{ for simplicity.}} \sin\Theta = 0$$

Useful trick: expand any nonlinear fcts in their Taylor series first.

$$\ddot{\Theta} + \Theta - \frac{1}{3!}\Theta^3 + \frac{1}{5!}\Theta^5 + \ldots = 0$$

$$\underbrace{\varepsilon\ddot{\Theta}_1 + \varepsilon^2\ddot{\Theta}_2 + \varepsilon^3\ddot{\Theta}_3 + \ldots}_{\ddot{\Theta}} +$$

$$\underbrace{\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \ldots}_{\Theta} +$$

$$-\frac{1}{3!}\left(\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \ldots\right)^3$$

$$+\frac{1}{5!}\left(\ldots\right)^5 + \ldots = 0$$

Aside:

$$\Theta^3 = (\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \dots)$$
$$(\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \dots)$$
$$(\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \dots)$$
$$= \varepsilon^3\Theta_1^3 + 3\varepsilon^4\Theta_1^2\Theta_2 + \dots$$

Into ODE:

$$\varepsilon\ddot{\Theta}_1 + \varepsilon^2\ddot{\Theta}_2 + \varepsilon^3\ddot{\Theta}_3 + \dots$$
$$\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \dots$$
$$-\frac{1}{6}(\varepsilon^3\Theta_1^3 + 3\varepsilon^4\Theta_1^2\Theta_2 + \dots)$$
$$+ \dots = 0$$

Collect powers of ε:

ε^1: $\ddot{\Theta}_1 + \Theta_1 = 0$

ε^2: $\ddot{\Theta}_2 + \Theta_2 = 0$

ε^3: $\ddot{\Theta}_3 + \Theta_3 = \frac{1}{6}\Theta_1^3$

Expand ICs too:

$$\Theta(t=0) = \varepsilon = \varepsilon\Theta_1(t=0) + \varepsilon^2\Theta_2(t=0) + \dots$$

$$\dot{\Theta}(t=0) = 0 = \varepsilon\dot{\Theta}_1(t=0) + \varepsilon^2\dot{\Theta}_2(t=0) + \dots$$

Hierarchy of IVPs:

ε^1:
$$\ddot{\Theta}_1 + \Theta_1 = 0$$
$$\Theta_1(t=0) = 1$$
$$\dot{\Theta}_1(t=0) = 0$$
$$\Rightarrow \Theta_1(t) = \cos(t)$$

ε^2:
$$\ddot{\Theta}_2 + \Theta_2 = 0$$
$$\Theta_2(t=0) = 0$$
$$\dot{\Theta}_2(t=0) = 0$$
$$\Rightarrow \Theta_2(t) = 0$$

ε^3:
$$\ddot{\Theta}_3 + \Theta_3 = \frac{1}{6}\Theta_1^3 = \frac{1}{6}\cos^3(t)$$
$$\Theta_3(t=0) = 0$$
$$\dot{\Theta}_3(t=0) = 0$$

$$\cos^3(t) = \frac{1}{4}\left(\cos(3t) + 3\cos(t)\right)$$

$$\ddot{\Theta}_3 + \Theta_3 = \frac{1}{24}\left(\cos(3t) + 3\cos(t)\right) \quad (6$$

cos(t)
solves
homog ODE,
then
sin(t) appears
⋮
apply ICs:

$$\Theta_3(t) = \frac{1}{192}\left(\cos(t) - \cos(3t)\right) + \frac{1}{16} t \sin(t)$$

[Numerical] experiment: Finite-amplitude oscillation of an undamped pendulum

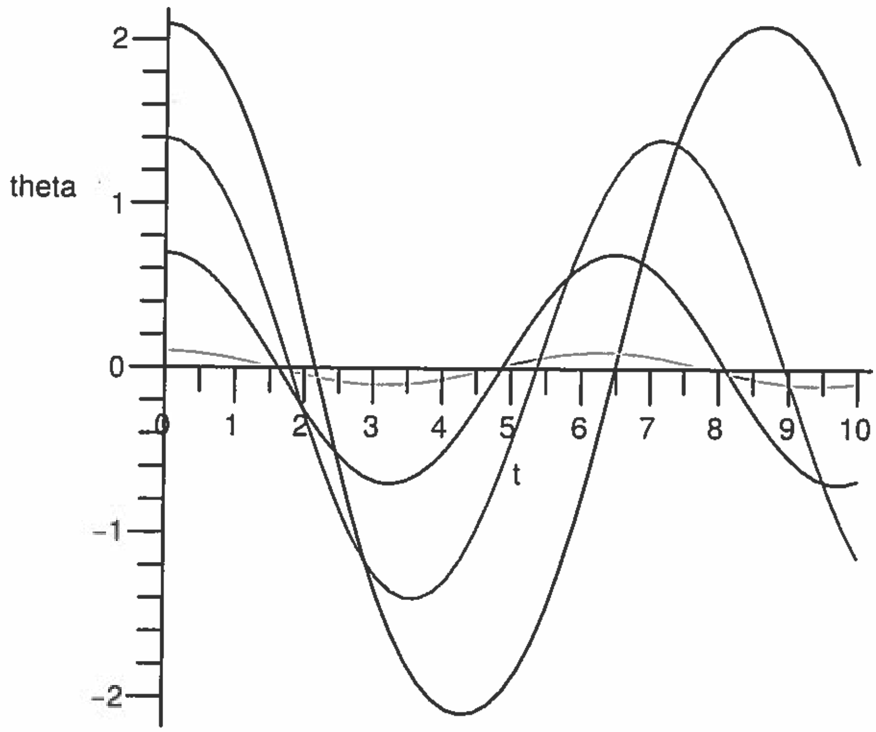
- Governing (non-linear!) ODE:

$$\ddot{\theta} + \sin\theta = 0$$

subject to the initial conditions

$$\theta(t=0) = \epsilon \quad \text{and} \quad \dot{\theta}(t=0) = 0.$$

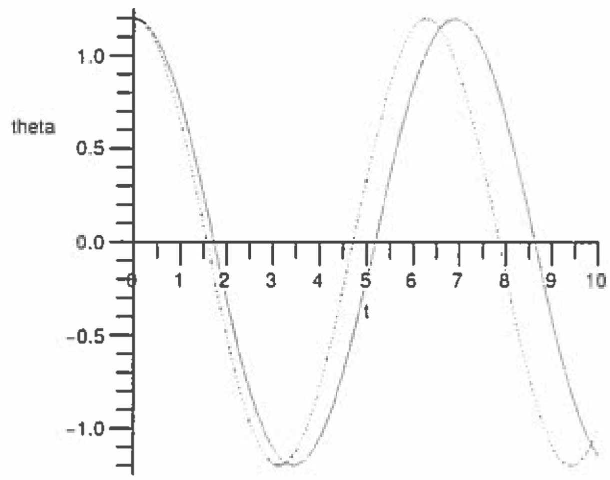
- Plot for $\epsilon = 0.1, 0.7, 1.4, 2.1$:



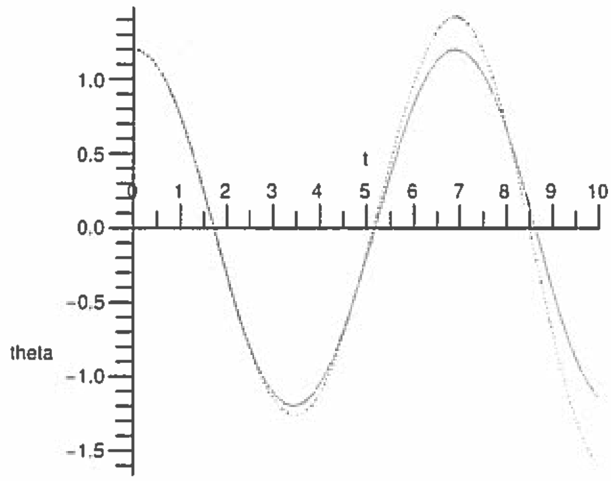
- **Observation:** Period of the oscillation increases for larger amplitudes.

Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$

- One-term perturbation solution (red), exact solution (green):

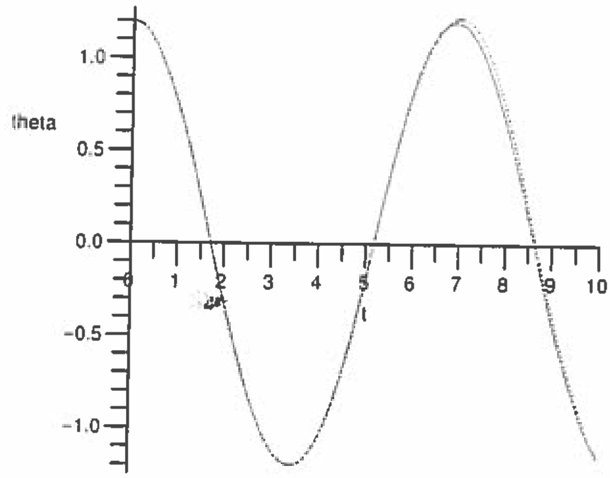


- Two-term perturbation solution (red), exact solution (green):

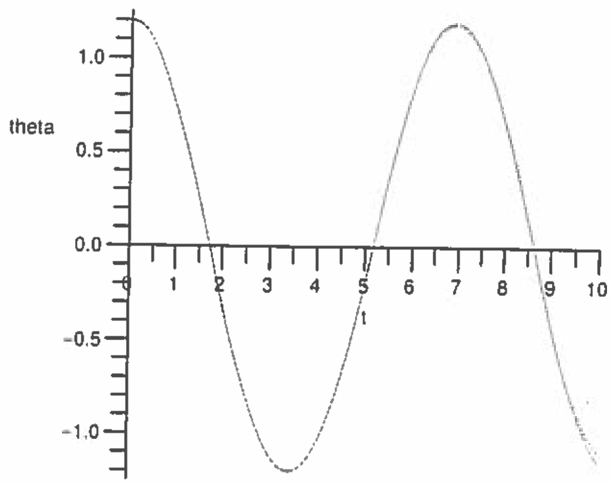


Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$ (cont.)

- Three-term perturbation solution (red), exact solution (green):

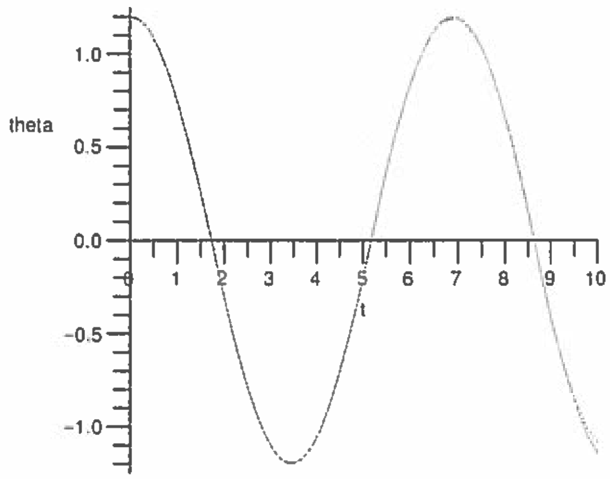


- Four-term perturbation solution (red), exact solution (green):

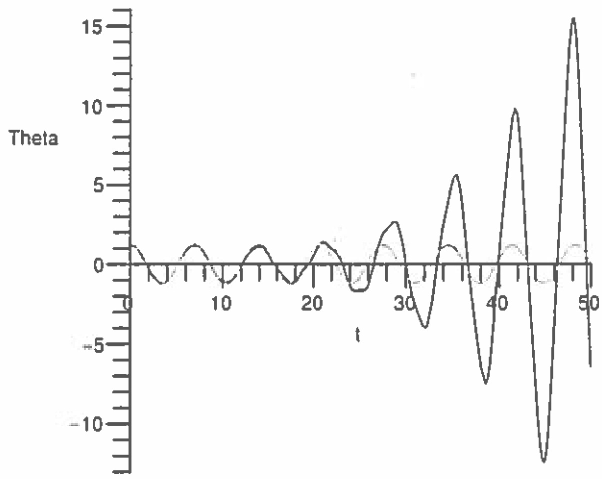


Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$ (cont.)

- Four-term perturbation solution (red), exact solution (green):



- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:



“Multinomial expansions”

- One tedious task that one tends to face regularly when using perturbation methods is that of raising a power series in ϵ to some integer power

$$S = \left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + ...\right)^n, \tag{1}$$

and collecting the terms multiplied by the same power of ϵ, i.e. re-writing S in the form

$$S = S_0(x_0) + \epsilon\, S_1(x_0, x_1) + \epsilon^2\, S_2(x_0, x_1, x_2) + ... \tag{2}$$

where the functions $S_i(x_0, x_1, ...)$ do not depend on ϵ.

- Formally, the expansion of S may be obtained by using the “multinomial series” (a generalisation of the binomial series) as

$$\left(a_1 + a_2 + ... + a_k\right)^n = \sum_{\substack{n_1, n_2, n_3, ..., n_k \in \mathbb{N}_0 \\ n_1 + n_2 + ... + n_k = n}} \frac{n!}{n_1!\, n_2! \dots n_k!}\, a_1^{n_1}\, a_2^{n_2} \dots a_k^{n_k}$$

see, e.g. `http://mathworld.wolfram.com/MultinomialSeries.html`

- However, we usually only need the first few terms in (2) for low-ish powers of n. Here they are:

$$\begin{aligned}
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + ...\right)^2 &= \left(x_0^2\right) + \epsilon\left(2x_0\, x_1\right) + \epsilon^2\left(x_1^2 + 2x_0\, x_2\right) + ... \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + ...\right)^3 &= \left(x_0^3\right) + \epsilon\left(3x_0^2\, x_1\right) + \epsilon^2\left(3x_0 x_1^2 + 3x_0^2\, x_2\right) + . \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + ...\right)^4 &= \left(x_0^4\right) + \epsilon\left(4x_0^3\, x_1\right) + \epsilon^2\left(4x_0^3 x_2 + 6x_0^2\, x_1^2\right) + .
\end{aligned}$$

- **Exercise:** Convince yourself that you understand how these terms arise. **Hint:** Either use the multinomial series given above, or write S explicitly as a product of n power series [e.g. for $n = 2:\ \ S = (x_0 + \epsilon\, x_1 + ...)(x_0 + \epsilon\, x_1 + ...)$] and inspect which combination of terms gives rise to what powers of ϵ.

- **Relax!** In an exam these expressions would be provided if $n > 2$.