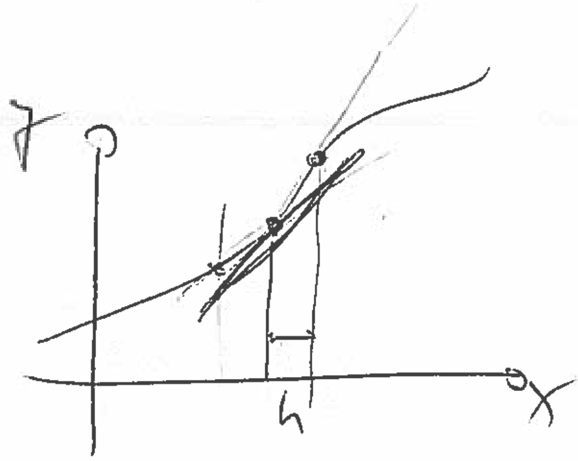
INTRODUCTION

Notation, Definitions and "What are the issues?"

The Derivative



Given a function

$$y(x)$$

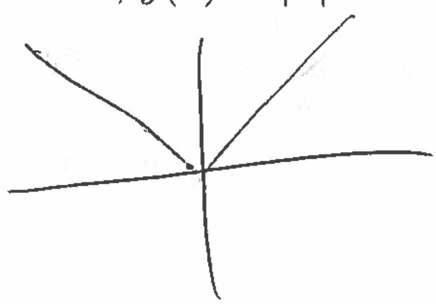
where

- x is the independent variable,
- y is the dependent variable,

the derivative is defined as

$$\begin{aligned} y'(x) = \frac{dy}{dx} &= \lim_{h\to 0} \frac{y(x+h)-y(x)}{h} \\ &= \lim_{h\to 0} \frac{y(x)-y(x-h)}{h}. \end{aligned}$$

The derivative is not defined at points where the "right" and "left" limits do not converge to the same value. For instance, $y(x) = |x|$ does not have a derivative at $x = 0$.



Higher Derivatives

Higher derivatives are defined recursively

$$\begin{aligned} y''(x) &= \frac{d^2y(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy(x)}{dx}\right) \\ y'''(x) &= \frac{d^3y(x)}{dx^3} = \frac{d}{dx}\left(\frac{d^2y(x)}{dx^2}\right) \\ &\text{etc.} \end{aligned}$$

...provided the lower-order derivatives are sufficiently smooth for the higher derivatives to exist.

Notation

- Dash notation:

$$\frac{d}{dx}(.) = (.)'$$
$$\frac{d^2}{dx^2}(.) = (.)''$$
$$\frac{d^n}{dx^n}(.) = (.)^{(n)}$$

- Dot notation: For time-dependent problems, where t is the independent variable, dots are often used to indicate derivatives.

$$x(t)$$
$$\frac{dx}{dt} = \dot{x}(t)$$
$$\frac{d^2x}{dt^2} = \ddot{x}(t)$$

- The dependence on the independent variable may be suppressed. For instance, instead of

$$y'(x) + p(x)\,y(x) = r(x)$$

we can simply write

$$y' + p(x)\,y = r(x)$$

because it's "obvious" that y is a function of x.

Ordinary differential equations

Definition:

- An n-th order ordinary differential equation (ODE) for $y(x)$ has the general form

$$\mathcal{F}\big(x, y(x), y'(x), ..., y^{(n)}(x)\big) = 0, \tag{1}$$

  i.e. it relates the (unknown) function, $y(x)$ to x and its 1st, 2nd, ..., nth derivatives.

- Often the implicit form given above can be solved for $y(x)$, allowing the ODE to be written in explicit form as:

$$y^{(n)}(x) = f\big(x, y(x), y'(x), ..., y^{(n-1)}(x)\big). \tag{2}$$

Solutions:

- A solution of the ODE (1) [or (2)] in an interval

$$I = \{x \mid a < x < b\}$$

  is *any* function $\phi(x)$ for which

$$\mathcal{F}\big(x, \phi(x), \phi'(x), ..., \phi^{(n)}(x)\big) = 0 \qquad \forall x \in I. \tag{3}$$

Notes:

- The statement already suggests that there may be multiple solutions.
- Furthermore, solutions might not exist for all values of x.
- In fact, there might not be a solution at all!

Two properties worth looking out for...

1. Linearity

- An ODE is linear if

$$\mathcal{F}\big(x, y(x), y'(x), ..., y^{(n)}(x)\big) = 0,$$

is linear in y and all its derivatives.

- Linear ODEs can be written as

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y = b(x)$$

where $a_i(x)$ $(i = 0, ..., n)$ and $b(x)$ are given functions.

2. Autonomous ODEs

- An ODE is autonomous if it has the form

$$\mathcal{F}\big(y(x), y'(x), ..., y^{(n)}(x)\big) = 0,$$

i.e. if the independent variable, x, does not appear explicitly.

Examples:

Ex: $y'' - \tan x \, y' = -2\sin x$

is 2nd order, linear non-autonomous ODE for $y(x)$.

$y(x) = \sin(x)$

check:

$y' = \cos(x)$

$y'' = -\sin(x)$

$y'' - \tan x \, y' \overset{?}{=} -2\sin x$

$-\sin(x) - \frac{\sin(x)}{\cancel{\cos(x)}} \cancel{\cos(x)} \overset{?}{=} -2\sin x$ ✓

Are there other solns. ?

Ex: $\frac{1}{24} y''' + y^{1/4} = 2x$

is 3^{rd} order, nonlinear, non-autonomous ODE for $y(x)$.

$y(x) = x^4$

Check: $y''' = 4 \cdot 3 \cdot 2\, x$

$y^{1/4} = x$

into ODE:

$\frac{1}{24} y''' + y^{1/4} \stackrel{?}{=} 2x$

$x + x \stackrel{?}{=} 2x$ ✓

Are there other solns?

Ex: $\frac{1}{24} y''' - y^{1/4} = 0$

is 3^{rd} order, nonlinear, autonomous ODE for $y(x)$.

$y(x) = x^4$ solves this one too.

other solns? Yes: $y = 0$.

So: Easy to check that a candidate soln. is a soln. How do we do this systematically?

- Existence? Are there any solutions?
- Uniqueness? Is there only one soln.?

Existence:

If the ODE describes a physical problem (and if it does so properly) then we expect the solution to exist.

However, existence is not obvious:

Ex: $$y' + \frac{1}{y'} = 0$$

is 1st order, nonlin, autonomous ODE for $y(x)$.

$$(y')^2 = -1$$

$$y' = \sqrt{-1} = i$$

Does not have a real soln.

Uniqueness:

Ex: $\dfrac{d^2y}{dx^2} = 0$

2nd order, linear, auton. ODE for $y(x)$

Integrate twice:

$$y(x) = Ax + B$$

arbitrary constants

Observation: If we want to achieve uniqueness we have to apply two additional constraints

E.g.: $\left.\begin{aligned} y(0) &= 0 \\ \left.\frac{dy}{dx}\right|_{x=0} &= 3 \end{aligned}\right\} \; y(x) = 3x$

$\left.\begin{aligned} y(0) &= 1 \\ y(1) &= 10 \end{aligned}\right\} \; y(x) = 9x + 1$

Boundary and initial conditions

> An m-th order ODE must be augmented by m constraints (in the form of "boundary" or "initial" conditions) if there is to be a unique solution.

Notes:

- This is a necessary, not a sufficient condition: Even if an m-th order ODE is augmented by m constraints, there may be multiple (or no!) solutions.

Initial conditions (ICs)

- If all constraints are applied at the same value of the independent variable, we refer to them as *initial conditions.*

Boundary conditions (BCs)

- If the constraints are applied at multiple values of the independent variable (typically at the ends of the interval I in which the solution is sought), we refer to them as *boundary conditions.*

Boundary and initial value problems

Initial value problems (IVPs)

"IVP = ODE + ICs"

- Initial value problems (IVPs) typically describe evolution processes in which the initial state (at time $t = t_0$, say) of a system is characterised by the initial conditions, while the ODE describes the dynamics of its subsequent evolution.

Boundary value problems (BVPs)

"BVP = ODE + BCs"

- Boundary value problems (BVPs) typically describe spatial problems in which the boundary conditions describe the state of the system on the domain boundaries, while the ODE governs its behaviour in the interior of the domain.

Examples: IVP

1D motion of a particle of mass m, subject to a force F(t) is governed by Newton's law:

$$m\frac{d^2x}{dt^2} = F(t)$$ 2nd order linear ODE for $x(t)$

IC: $x(t=0) = x_0$ initial posn.

$$\left.\frac{dx}{dt}\right|_{t=0} = v_0$$ initial veloc.

Existence & uniqueness?

"obvious" (physically)

Special case:

$$F(t) = F_0 = \text{const}$$

integrate twice:

$$x(t) = \frac{1}{2}\frac{F_0}{m}t^2 + At + B$$

arbitrary constants

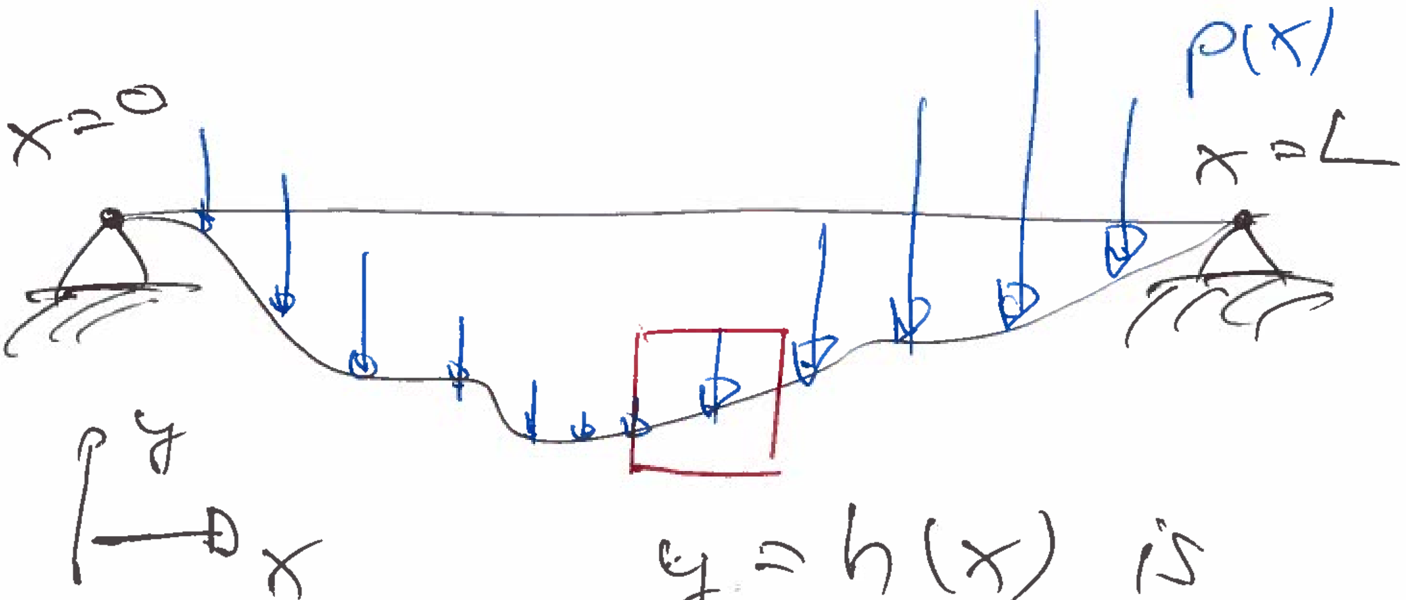
Apply IC:

$$x(t) = \frac{1}{2}\frac{F_0}{m}t^2 + v_0 t + x_0$$

unique!

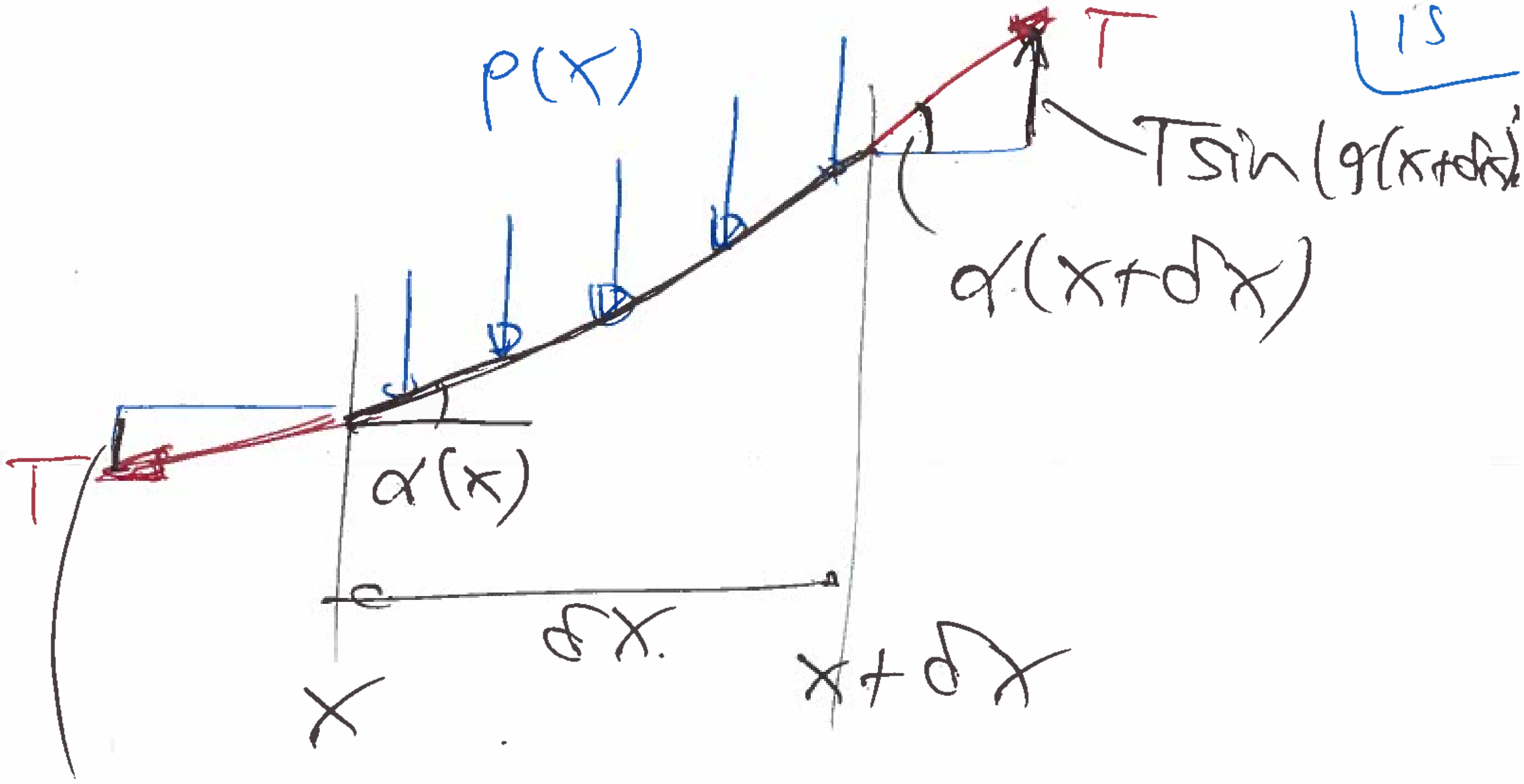
Example: BVP

Elastic string under constant tension T loaded by a transverse pressure $p(x)$.



$y = h(x)$ is the position of the string.

Balance of forces on small segment of string:



$T\sin(\alpha(x))$

$$\underbrace{p(x)\,dx}_{\text{downward force from pressure}} = \underbrace{T\sin(\alpha(x+dx)) - T\sin(\alpha(x))}_{\text{net upward force from tension.}}$$

Limits: • $dx \to 0$

• Small slopes only $|\alpha| \ll 1$

$|\alpha| \ll 1$:

$$\sin\alpha \approx \tan\alpha = \frac{dh}{dx}$$

$$\rho(x)\,\delta x = T\left(\left.\frac{dh}{dx}\right|_{x+\delta x} - \left.\frac{dh}{dx}\right|_{x}\right)$$

$$\rho(x) = \underbrace{\lim_{\delta x \to 0} T\,\frac{\left(\left.\frac{dh}{dx}\right|_{x+\delta x} - \left.\frac{dh}{dx}\right|_{x}\right)}{\delta x}}_{T\frac{d}{dx}\left(\frac{dh}{dx}\right)}$$

$$\boxed{T\,\frac{d^2h}{dx^2} = \rho(x)}$$

compare:

$$\boxed{m\,\frac{d^2x}{dt^2} = F(t)}$$