Perturbation methods

$$x^2 + \varepsilon x - 1 = 0$$

$$x(\varepsilon) = -\frac{1}{2}\varepsilon \pm \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + 1}$$

has Taylor expansion:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

from known solution!

Idea: If we don't know soln: use power series as ansatz.

$\Rightarrow$ Into eqn.
- collect powers of ε
- set coeffns to zero

$\Rightarrow$ Hierarchy of eqns for $x_0, x_1, \dots$

$\Rightarrow$ eqn satisfied to increasing accuracy in ε.

An ODE example:

Damped oscillator with small damping.

$$\boxed{\begin{aligned} &\ddot{x} + \varepsilon \dot{x} + x = 0 \\ &x(0) = 1 \\ &\dot{x}(0) = 0 \end{aligned}} \qquad 0 \le \varepsilon \ll 1$$

Assume we don't know the solution. But note that problem is easy for $\varepsilon = 0$.

Ansatz:

$$\cancel{x(t)}\; x(t;\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$$

$$\dot{x}(t) = \dot{x}_0(t) + \varepsilon \dot{x}_1(t) + \varepsilon^2 \dot{x}_2(t) + \ldots$$

$$\ddot{x}(t) = \ddot{x}_0(t) + \varepsilon \ddot{x}_1(t) + \varepsilon^2 \ddot{x}_2(t) + \ldots$$

into ODE

$$\underbrace{\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots}_{\ddot{x}} +$$

$$\underbrace{\varepsilon \dot{x}_0 + \varepsilon^2 \dot{x}_1 + \varepsilon^3 \dot{x}_2 + \dots}_{\varepsilon \dot{x}} +$$

$$\underbrace{x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots}_{x} = 0$$

collect powers of ε:

$$(\ddot{x}_0 + x_0)\,\varepsilon^0 +$$
$$(\ddot{x}_1 + \dot{x}_0 + x_1)\,\varepsilon^1 +$$
$$(\ddot{x}_2 + \dot{x}_1 + x_2)\,\varepsilon^2 + \dots = 0$$

IC:

$$x(0) = \underline{x_0(0)} + \varepsilon x_1(0) + \dots = \underline{1}$$
$$\dot{x}(0) = \underline{\dot{x}_0(0)} + \varepsilon \dot{x}_1(0) + \dots = 0$$

Now collect like powers of ε from ODE & ICs

$\underline{\varepsilon^0:}$

$$\boxed{\begin{aligned} \ddot{x}_0 + x_0 &= 0 \\ x_0(0) &= 1 \\ \dot{x}_0(0) &= 0 \end{aligned}}$$

$\underline{\varepsilon^1:}$

$$\boxed{\begin{aligned} \ddot{x}_1 + x_1 &= -\dot{x}_0 \\ x_1(0) &= 0 \\ \dot{x}_1(0) &= 0 \end{aligned}}$$

$\underline{\varepsilon^2:}$

$$\boxed{\begin{aligned} \ddot{x}_2 + x_2 &= -\dot{x}_1 \\ x_2(0) &= 0 \\ \dot{x}_2(0) &= 0 \end{aligned}}$$

$\vdots$

Same structure as in the algebraic example:

Leading-order problem (for $\varepsilon = 0$) is exactly the full problem when we set $\varepsilon = 0$. We had assumed that this was easier / do-able.

The corrections $x_1, x_2, \dots$ come from a hierarchy of problems that involve the known previous solns.

$\underline{\varepsilon^0:}$ $\ddot{x}_0 + x_0 = 0$ $\quad x_0(0) = 1$

$\dot{x}_0(0) = 0$

$x_0(t) = \cos(t)$

$\underline{\varepsilon^1:}$ $\boxed{\ddot{x}_1 + x_1 = -\dot{x}_0} = \boxed{\sin(t)}$

$x_{1_H} = A\cos t + B\sin t$

$x_{1p} = \,?$ $\quad$ Note $x_{1p} = C\sin(t)$ won't work.

$\hat{x}_{1p} = Ct\sin(t)$ produces $\cos(t)$ $\Rightarrow$ add this

$x_{1p} = Ct\sin(t) + Dt\cos(t)$

into ODE ... $\to$ EXERCISE

$C = 0$ & $D = -1/2$

$$x_1(t) = A\cos(t) + B\sin(t) - \frac{1}{2}t\cos(t)$$

Apply IC: ...

$A = 0$ $B = 1/2$

$$x_1(t) = \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t).$$

$\underline{\varepsilon^2:}$ $\ddot{x}_2 + x_2 = \underbrace{-\dot{x}_1}_{-\frac{1}{2}t\sin t}$ $x_2(0) = \dot{x}_2(0) = 0$

$$x_{p_2} = -\frac{1}{8}t\sin(t) + \frac{1}{8}t^2\cos t$$

(EXERCISE)

This happens to satisfy the ICs so: $x_{p_2} = x_2(t)$

ETC

Mechanical oscillator with weak damping

- Governing (linear) ODE:

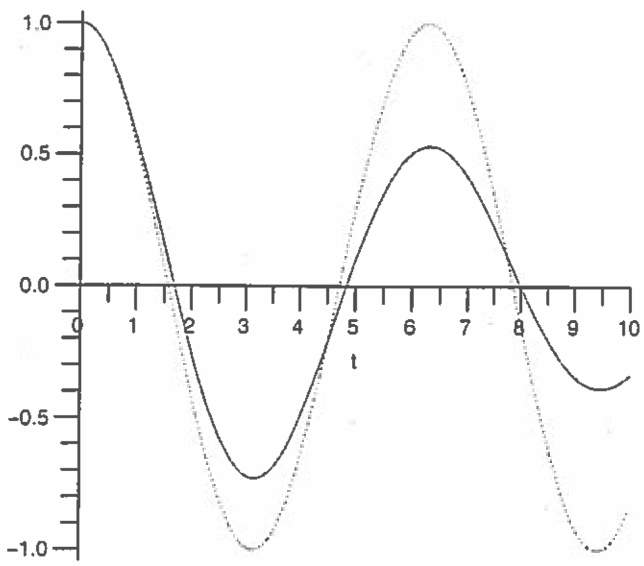
$$\ddot{x} + \epsilon \dot{x} + x = 0$$

subject to the initial conditions

$$x(t=0) = 1 \quad \text{and} \quad \dot{x}(t=0) = 0.$$

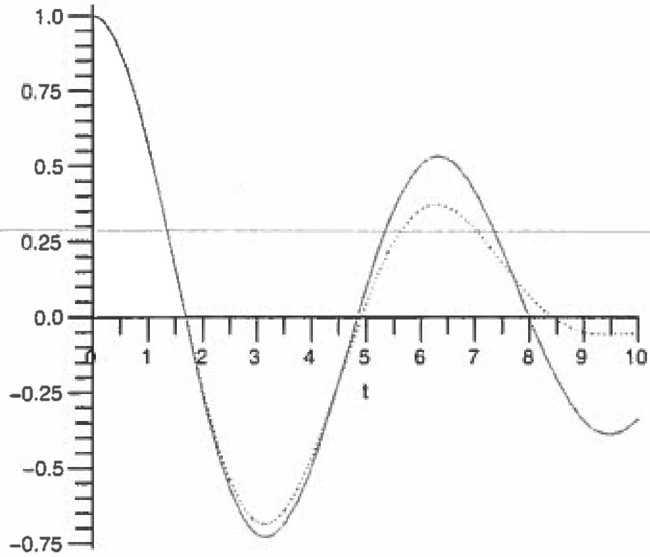
Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- One-term perturbation solution (red), exact solution (green):

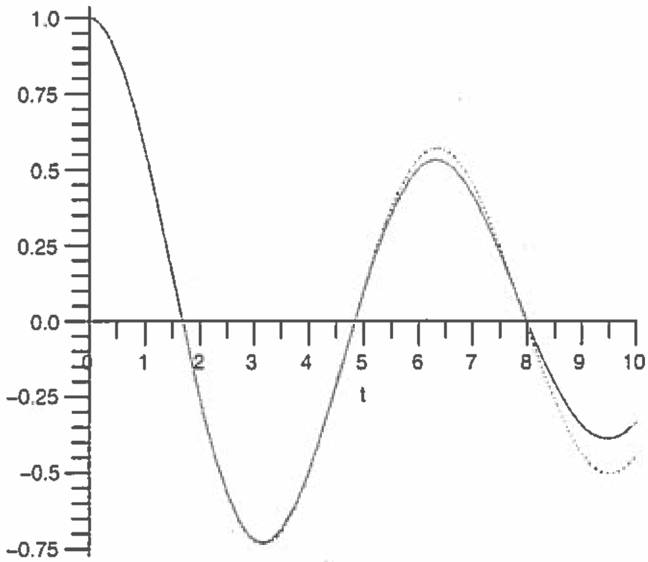


Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Two-term perturbation solution (red), exact solution (green):

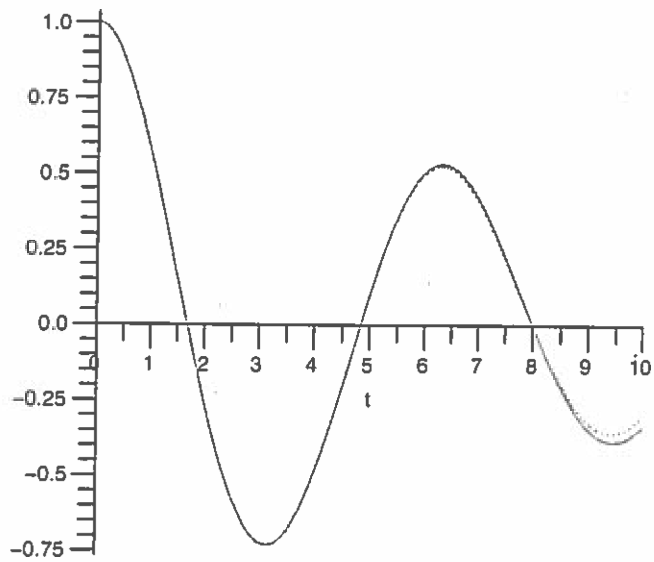


- Three-term perturbation solution (red), exact solution (green):

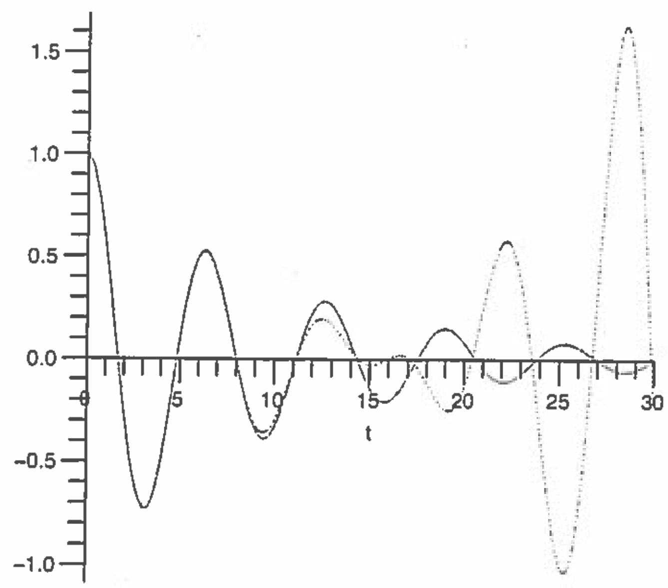


Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Four-term perturbation solution (red), exact solution (green):



- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:



Observations:

1. At fixed time the perturbation solution converges rapidly to the exact soln as:

$$\varepsilon \to 0$$

and/or the number of terms in the expansion is increased.

2. As $t \to \infty$ the solution becomes increasingly inaccurate
   - Symptom: t^n terms grow
   - Underlying cause: Errors in satisfying the ODE only approximately accumulate.

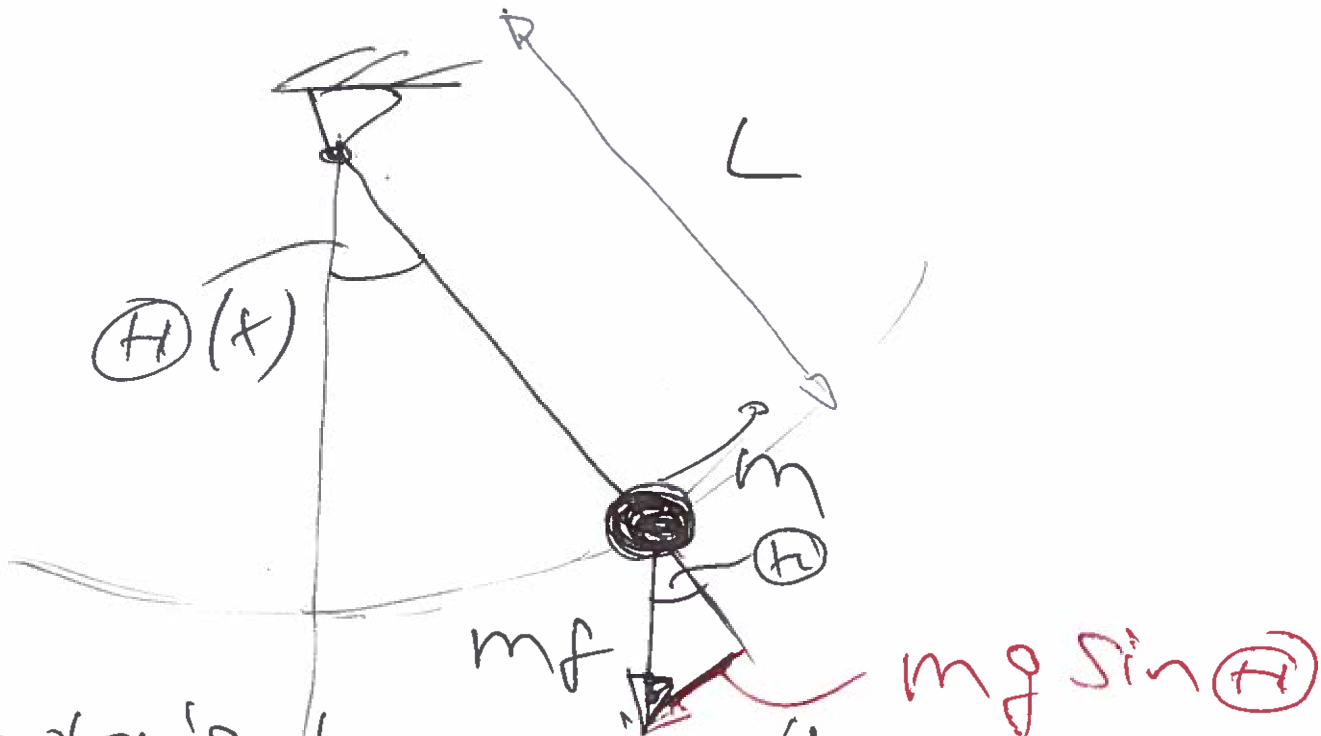
3. For any fixed time interval $0 \leq t \leq T$ we can choose ε or the number

dt terms to make the [P

soln. arbitrarily accurate.

A nonlinear example

Pendulum:



Newton's law in the tangential direction

$$\cancel{m}\frac{d}{dt}\Big(\underbrace{L\frac{d\Theta}{dt}}_{\text{velocity in tangential direction}}\Big) = -\cancel{m}g\sin\Theta$$

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$$\ddot{\Theta} + \omega^2 \sin\Theta = 0$$

where $\omega^2 = \frac{g}{L}$

<u>IC:</u> $\Theta(t=0) = \varepsilon$

$\dot{\Theta}(t=0) = 0$

[Numerical] experiment: Finite-amplitude oscillation of an undamped pendulum

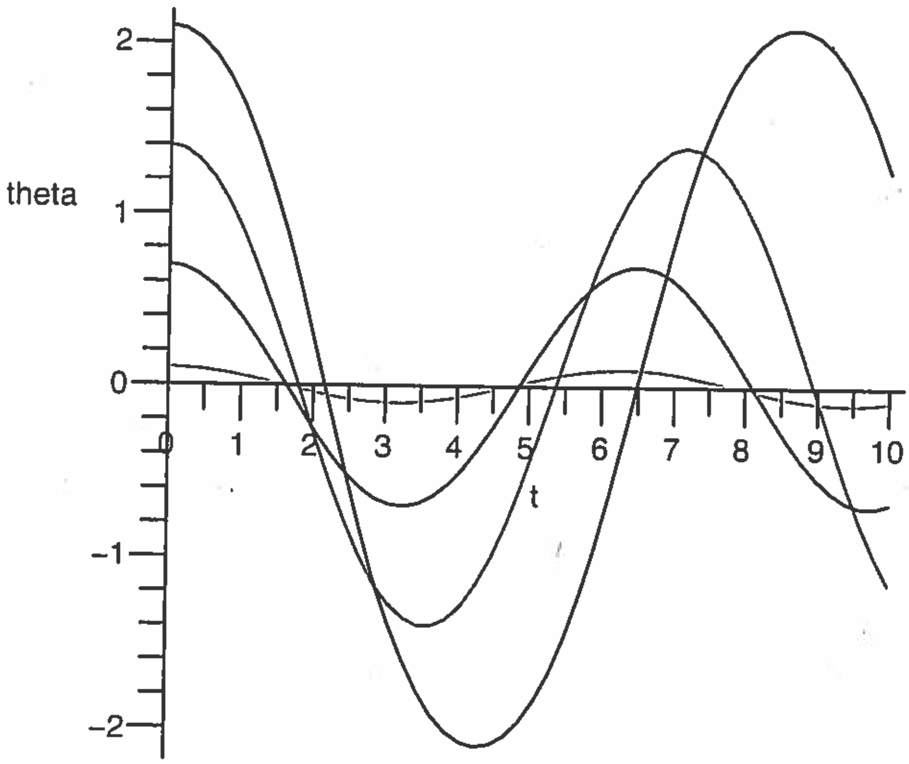
- Governing (non-linear!) ODE:

$$\ddot{\theta} + \sin\theta = 0$$

subject to the initial conditions

$$\theta(t=0) = \epsilon \quad \text{and} \quad \dot{\theta}(t=0) = 0.$$

- Plot for $\epsilon = 0.1, 0.7, 1.4, 2.1$:



- **Observation:** Period of the oscillation increases for larger amplitudes.