Week 7 Coursework Test (2018) | *math-10222/11222* | Calculus and Applications

Date, Time, Place, Duration and Type of Test:

Monday, March 12th 2018, 9-9.50am.

- **Surnames starting with A-K:** Crawford House, Lecture Theatre A
- **Surnames starting with L-Z:** Kilburn, Lecture Theatre 1.1

The test will last 50 minutes.

You will be asked to present your answers as "***multiple choice***" responses :

- make sure that you select only one answer to each question
- blank answers or more than one answer to any one question will be given zero marks
- wrong answers will be given a negative mark, ***so avoid guessing any answers!***

What to bring:

Pens, pencils and an eraser.

Your student ID card.

Be sure that you know the name of the person taking your Calculus and Applications supervision class each week (you will be asked to provide the name of your supervisor who will return the marked test to you).

What not to bring:

No calculators, paper, mobile phones, radios, MP3-players, books, bags, etc.

What you will be supplied with:

Question sheets and a multiple-choice answer sheet for providing your answers.

You will hand in only the answer sheet at the end of the test.

There will be space on the question sheets for working out your answers (not to be handed in).

What you will not be supplied with:

No formula tables will be provided for use in this test, so make sure that you know all relevant information, theorems, methods and formulae.

What to revise:

Test questions may cover any material up (and including) example sheet 4.

All of these example sheets and their sample answers will have been provided online.

If you are fully able to tackle the example sheets you should have no difficulty with the test.

You should use the problem sheets and the examples done in class for revision.

Marks for the test:

This test will count for 5% of your overall grade (10% for the Maths and Physics students).

Revise well and good luck!

$$F(x;\varepsilon)=0 \Longrightarrow x(\varepsilon)$$

ε small

$$x(\varepsilon)=x(0)+\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0}\varepsilon+\frac{1}{2!}\left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0}\varepsilon^2+\dots$$

suggests ansatz:

$$x(\varepsilon)=x_0+\varepsilon x_1+\varepsilon^2 x_2+\dots$$

unknown constants

Into $F(x;\varepsilon)$; expand & collect powers of ε:

$$\underbrace{F_0(x_0)}_{\text{solve for } x_0}+\varepsilon\underbrace{F_1(x_0,x_1)}_{\text{solve for } x_1}+\varepsilon^2\underbrace{F_2(x_0,x_1,x_2)}_{\text{solve for } x_2}+\dots=0$$

etc.

An ODE example

$$\boxed{\begin{aligned} &\ddot{x} + \varepsilon \dot{x} + x = 0 \\ &\text{IC: } x(t=0) = 1 \\ &\quad\;\; \dot{x}(t=0) = 0 \end{aligned}}$$

(weakly damped oscillator)

[<u>Note</u>: for $\varepsilon = 0$: $x(t) = \cos(t)$]

Assume we don't know soln for $\varepsilon \neq 0$.

want $x(t;\varepsilon)$.

<u>Ansatz</u>:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$$

$$\dot{x}(t) = \dot{x}_0(t) + \varepsilon \dot{x}_1(t) + \varepsilon^2 \dot{x}_2(t) + \ldots$$

$$\ddot{x}(t) = \ddot{x}_0(t) + \varepsilon \ddot{x}_1(t) + \varepsilon^2 \ddot{x}_2(t) + \ldots$$

into ODE

$$\underline{\ddot{x}_0} + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots +$$

$$\varepsilon \dot{x}_0 + \varepsilon^2 \dot{x}_1 + \varepsilon^3 \dot{x}_2 + \dots +$$

$$\underline{x_0} + \varepsilon x_1 + \varepsilon^2 x_2 + \dots = 0$$

collect powers of ε:

$$\boxed{(\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + \dot{x}_0 + x_1) + \varepsilon^2(\ddot{x}_2 + \dot{x}_1 + x_2) + \dots = 0}$$

IC:

$$x(t=0) = \underline{x_0(0)} + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \dots = \underline{1}$$

$$\boxed{(x_0(0) - 1) + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \dots = 0}$$

$$\dot{x}(t=0) = \boxed{\dot{x}_0(0) + \varepsilon \dot{x}_1(0) + \varepsilon^2 \dot{x}_2(0) + \dots = 0}$$

Now collect coefficients of powers of ε in ODE & ICs:

ε^0:

$$\ddot{x}_0 + x_0 = 0$$
$$x_0(0) = 1$$
$$\dot{x}_0(0) = 0$$

orig problem for $\varepsilon = 0$

$x_0 = \text{cost}$

ε^1:

$$\ddot{x}_1 + x_1 = -\dot{x}_0 \longrightarrow x_1$$
$$x_1(0) = 0$$
$$\dot{x}_1(0) = 0$$

ε^2:

$$\ddot{x}_2 + x_2 = -\dot{x}_1$$
$$x_2(0) = 0$$
$$x_2(0) = 0$$

ε^n:

$$\ddot{x}_n + x_n = -\dot{x}_{n-1}$$
$$x_n(0) = 0$$
$$\dot{x}_n(0) = 0$$

$\underline{\varepsilon^0:}$ $\ddot{x}_0 + x_0 = 0$

$x_0(t) = A\cos t + B\sin t$

$\underline{IC:}$ $x_0(0) = 1$ & $\dot{x}_0(0) = 0$

$\Rightarrow x_0(t) = \cos t$

$\underline{\varepsilon^1:}$ $\ddot{x}_1 + x_1 = -\dot{x}_0 = \sin t$

$x_1 = x_{1_H} + x_{1P}$

$x_{1H} = A\cos t + B\sin t$

$x_{1P} = C t\sin(t) + D t\cos t$

b/c soln. of hom of ODE

plug in ... (EXERCISE)

$C = 0 \qquad D = -\frac{1}{2}$

$x_1(t) = A\cos t + B\sin t - \frac{1}{2}t\cos t$

Apply IC:

$x_1(0) = 0$ & $\dot{x}_1(0) = 0$

$\Rightarrow A = 0, \quad B = 1/2$

$$x_1(t) = \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$

ε^2: $\quad \ddot{x}_2 + x_2 = -\dot{x}_1 = -\frac{1}{2}t\sin t$

$$x_{2p} = -\frac{1}{8}t\sin t + \frac{1}{8}t^2\cos t$$

This already satisfies ICs so $x_2 = x_{2p}$.

Mechanical oscillator with weak damping

- Governing (linear) ODE:

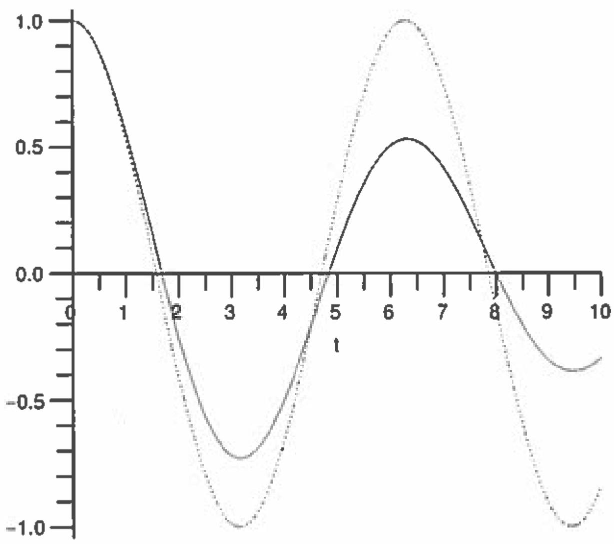
$$\ddot{x} + \epsilon\dot{x} + x = 0$$

subject to the initial conditions

$$x(t = 0) = 1 \quad \text{and} \quad \dot{x}(t = 0) = 0.$$

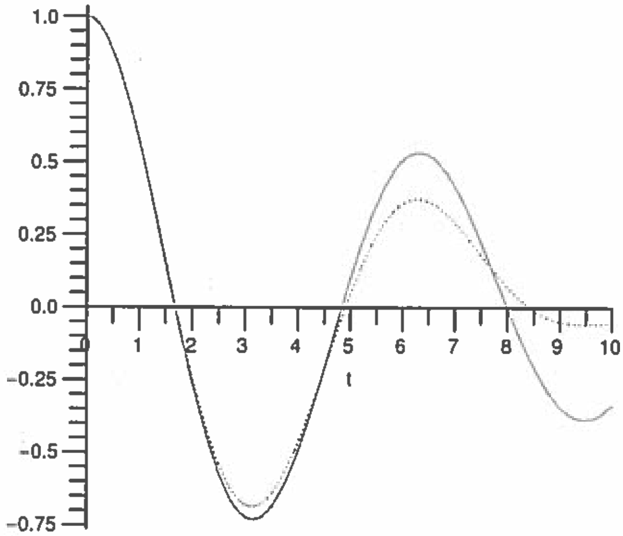
Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- One-term perturbation solution (red), exact solution (green):

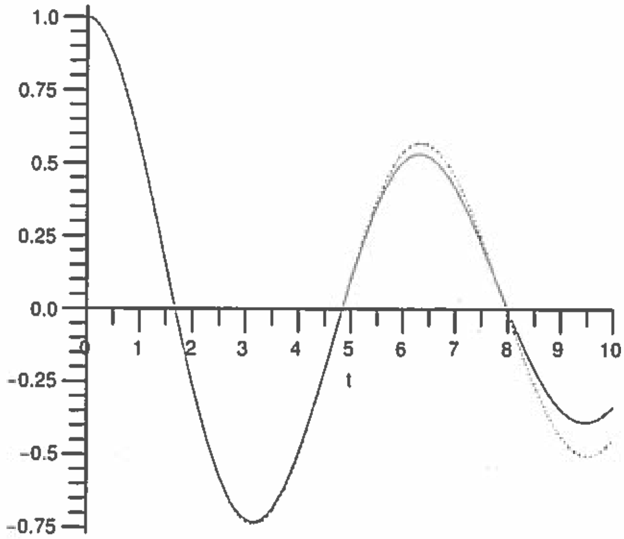


Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Two-term perturbation solution (red), exact solution (green):

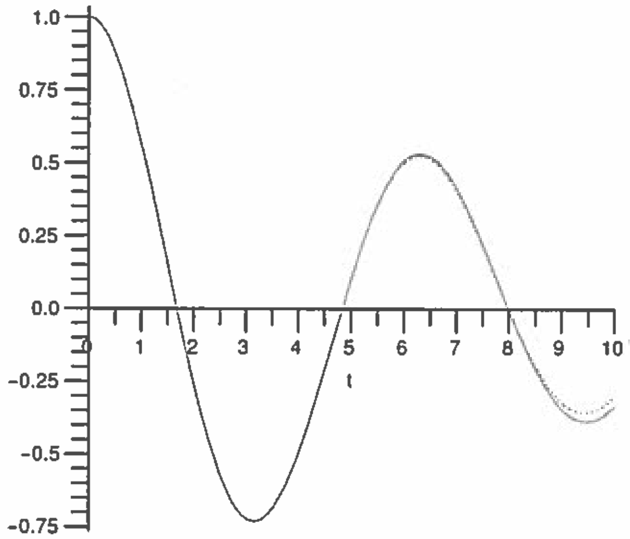


- Three-term perturbation solution (red), exact solution (green):

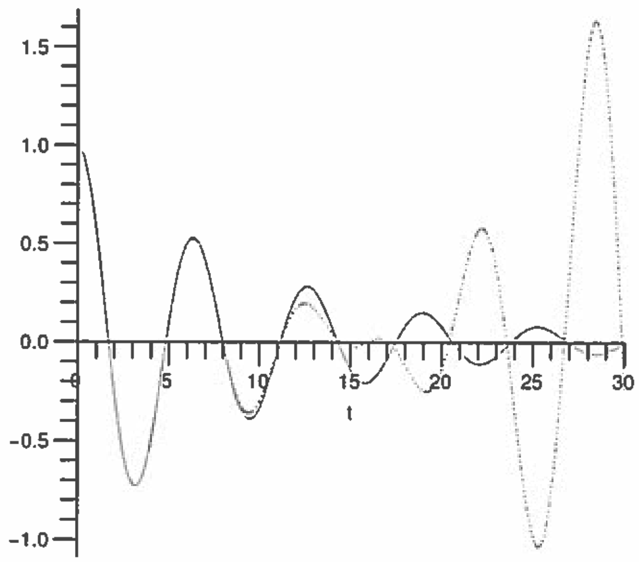


Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Four-term perturbation solution (red), exact solution (green):



- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:



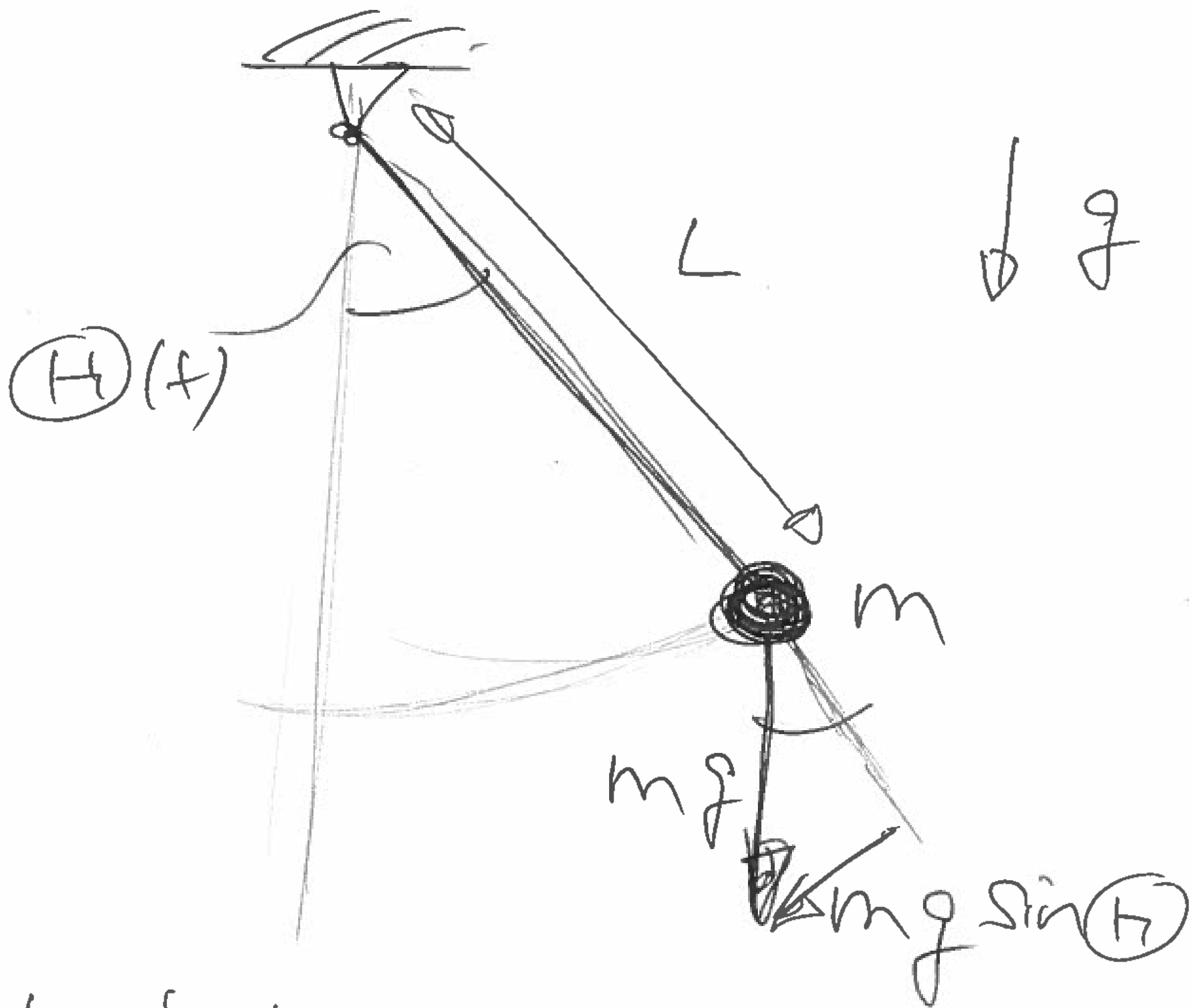
[See plots]

<u>Observations:</u>

1. Perturbation solution converges to exact solution over a finite time interval as:
   - the number of terms in expansion is increased
   - ε is decreased
2. As $t \to \infty$ the perturbation approx. becomes inaccurate because
   - t^n terms in the solution grow
   - Errors accumulate because ODE is only satisfied to a certain accuracy.

A nonlinear example

Pendulum:



Newton's law in tangential direction:

$$\boxed{\cancel{m}\frac{d}{dt}\left(L\frac{d\Theta}{dt}\right) = -\cancel{m}g\sin\Theta}$$

$$\frac{d^2\Theta}{dt^2} + \underbrace{\frac{g}{L}}_{\omega^2}\sin\Theta = 0$$

$$\frac{d^2\Theta}{dt^2} + \omega^2 \sin\Theta = 0$$

<u>IC:</u> $\Theta(t=0) = \varepsilon$

$$\left.\frac{d\Theta}{dt}\right|_{t=0} = 0$$

<u>nonlinear</u> 2^{nd} order ODE!

Existence and Uniqueness for *non-linear* 2nd order ODEs

Consider the *non-linear* second-order ODE

$$y'' = f(x, y, y') \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the function $f(x, y, y')$, are given.

Theorem

> If $f(x, y, y')$ and $\frac{\partial f(x,y,y')}{\partial y}$ and $\frac{\partial f(x,y,y')}{\partial y'}$ are continuous functions of x, y and y' in a region $0 < |x - X| < a$, $0 < |y - Y| < b$ and $0 < |y' - Z| < c$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

Notes:

- The statement is easily generalised to (even) higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

[Numerical] experiment: Finite-amplitude oscillation of an undamped pendulum

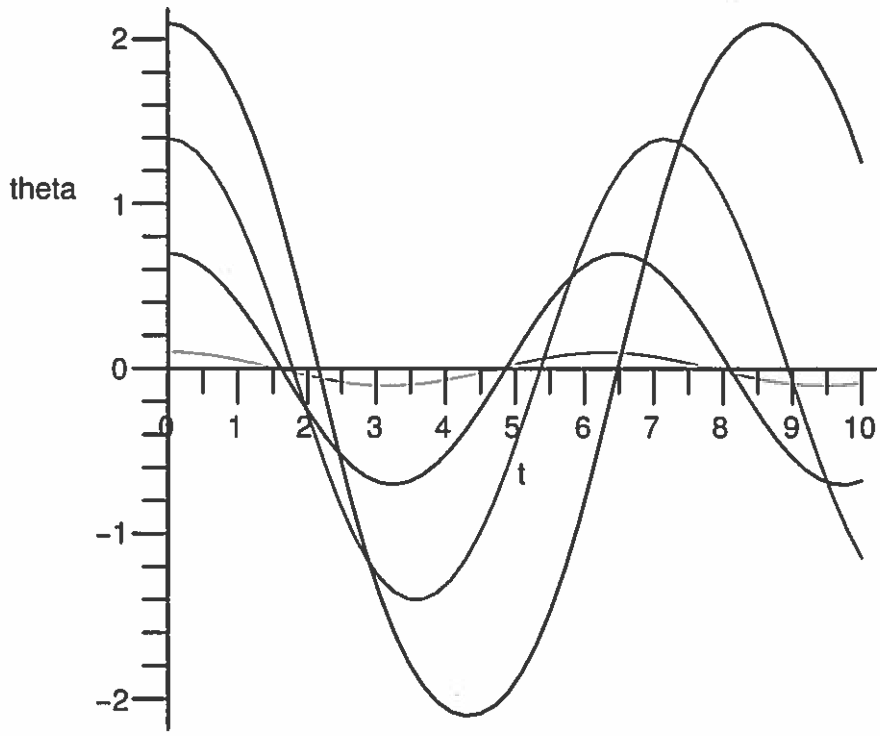
- Governing (non-linear!) ODE:

$$\ddot{\theta} + \sin\theta = 0$$

subject to the initial conditions

$$\theta(t=0) = \epsilon \quad \text{and} \quad \dot{\theta}(t=0) = 0.$$

- Plot for $\epsilon = 0.1, 0.7, 1.4, 2.1$:



- **Observation:** Period of the oscillation increases for larger amplitudes.