Forced osc:

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = \hat{f} e^{i\Omega t}$$

$$x(t) = e^{-\delta t}\left(A\cos\left(\sqrt{\omega^2-\delta^2}\,t\right) + B\sin\left(\sqrt{\omega^2-\delta^2}\,t\right)\right) + X e^{i\Omega t}$$

(real part only)

$$\frac{|X|}{|\hat{f}/\omega^2|} = \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2 + \left(2\left(\frac{\delta}{\omega}\right)\left(\frac{\Omega}{\omega}\right)\right)^2}}$$

Note: As $\delta \to 0$ $|X| \to \infty$ if $\Omega = \omega$: Resonance:

If $\delta = 0$

$$\ddot{x} + \omega^2 x = \hat{f}\cos(\Omega t) = \mathrm{Re}\left(\hat{f} e^{i\Omega t}\right)$$

Special case of the above analysis unless $\Omega = \omega$.

~~to~~ In that case the naive ansatz:

$$x_p = \bar{X} \cos(\Omega t)$$

is a soln. of the homog. ODE. Then:

$$x_p = \bar{X}\, t \cos(\Omega t) = \mathrm{Re}\left(\bar{X} t e^{i\Omega t}\right)$$

EXERCISE:

$$\bar{X} = -i \frac{\hat{f}}{2\Omega}$$

Then extract real part from

$$x(t) = \underbrace{A\cos(\Omega t) + B\sin(\Omega t)}_{\substack{\text{free oscillations} \\ \text{don't decay}}} + \frac{\hat{f}}{2\omega}\, t \sin(\Omega t)$$

Amplitude of the forced oscillation increases linearly with time.

Basic ideas of perturbation methods: "Exploiting small parameters" and "Scaling"

13

Observation 1:

- ODEs (and hence their solutions!) typically contain some parameters, e.g.

$$m\ddot{x} + k\dot{x} + cx = F\,\cos(\Omega t)$$

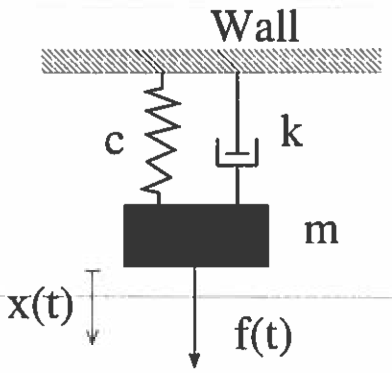
so

$$x = x(t) = x(t; m, k, c, \Omega).$$

- Often some of the problem's parameters are "small". How can we exploit this?
- Example:
  - Assume that we (only) know the solution of the above ODE for $k = 0$ (no damping).
  - What is the solution for "small" k?

Observation 2:

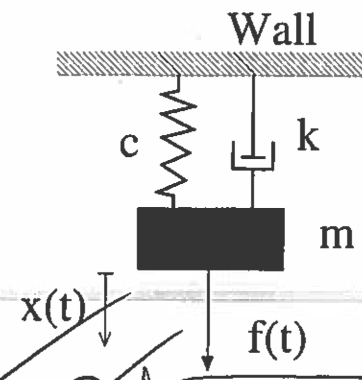
- ODEs that model physical phenomena typically express balances (of forces, energies, currents, ...).
- Here's an example of a balance of forces:



$$\underbrace{m\ddot{x}}_{\text{inertial forces}} + \underbrace{k\dot{x}}_{\text{damping forces}} + \underbrace{cx}_{\text{spring forces}} = \underbrace{F\cos(\Omega t)}_{\text{applied external force}}$$

- In general, all terms in the ODE will make a significant contribution to the overall "balance".
- However, there *may* be regimes in which the balance of terms is dominated by a balance between just a few (ideally two) terms, while the other terms only provide "negligible" contributions.
- The simplified equations (obtained by neglecting the small terms) are often much easier to solve than the full equations.
- We may [should!] then be interested in finding the effect that the "small" perturbations have on the solution.
- A seemingly trivial observation: You will need *at least* two terms to balance!

Example:



$$\overset{\sim\Omega^2}{\cancel{m\ddot{x}}} + \overset{\sim\Omega}{\cancel{k\dot{x}}} + \boxed{cx = F\cos(\Omega t)}$$

- We established earlier that
$$x(t) = x_P(t) + \cancel{x_H(t)}$$
where $x_H(t) \to 0$ very rapidly.

- Following the decay of the initial transients [described by $x_H(t)$] we have
$$x(t) \approx x_P(t) = \overset{F/c}{A}\cos(\Omega t) + \overset{0}{B}\sin(\Omega t)$$

- Hence if Ω is "small", the mass will move very slowly, implying that $m\ddot{x}$ and $k\dot{x}$ will be much smaller than cx.

- In this "quasi-steady" regime, we expect the motion of the mass to be described (approximately!) by
$$c\,x(t) \approx F\cos(\Omega t).$$

$$x(t) \approx \frac{F}{c}\cos(\Omega t)$$

“Proof”

- Check that
$$x(t) \approx \frac{F}{c}\cos(\Omega t)$$
is an approximate solution of
$$m\ddot{x} + k\dot{x} + cx = F\cos(\Omega t)$$
if Ω is small.

- The exact solution is
$$x(t) \approx x_P(t) = A\cos(\Omega t) + B\sin(\Omega t)$$
where
$$A = F\frac{c - \cancel{m\Omega^2}}{\cancel{(k\Omega)^2} + (c - \cancel{m\Omega^2})^2} \quad \to \frac{F}{c} \quad \text{as } \Omega \to 0,$$
and
$$B = F\frac{\cancel{k\Omega}}{\cancel{(k\Omega)^2} + (c - \cancel{m\Omega^2})^2} \quad \to 0 \quad \text{as } \Omega \to 0.$$

“Q.E.D.”

Observations about Observations 1, 2 and 3

- The approach outlined above exploits *additional* knowledge about the problem.
- You will either have such knowledge *a priori* or you can make certain (hopefully plausible) assumptions about certain properties of the solution.
- In the latter case, you'll have to check the consistency of your assumptions when you're done. For instance:
  - Assume the the solution is such that certain terms in the ODE are small.
  - Neglect the small terms in the ODE and solve.
  - Check afterwards that the terms that were *assumed* to be small are *actually* small.
- The approach tends to produce approximate solutions of the ODE that are valid only in certain *"regions of parameter space"*, e.g. for small forcing frequencies Ω, small damping k, etc.
- This is often more useful than having an exact (but horrendously complicated) closed-form solution that is valid for all parameter values.

Exploiting small parameters: (Regular) perturbation methods

An algebraic example:

$$x^2 + \varepsilon x - 1 = 0$$

$$x = -\frac{1}{2}\varepsilon \pm \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + 1} = x(\varepsilon)$$

$$0 \leq \varepsilon << 1$$

If ε is small can use Taylor expansion about $\varepsilon = 0$:

$$x(0) = \pm 1$$

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = \dots = -\frac{1}{2}.$$

$$\left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0} = \pm\frac{1}{4}$$

$$\vdots$$

$$x(\varepsilon) = x(\varepsilon=0) + \varepsilon \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} + \frac{1}{2!}\varepsilon^2 \left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0}$$

$$+ \cdots$$

$$x(\varepsilon) = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 \mp \frac{1}{128}\varepsilon^4 + \cdots$$

This Taylor series converges for $|\varepsilon| < 2$ but far more importantly: The first few terms give an excellent approximation if ε is small.

Observation:

Have exact solution $\longrightarrow$ expand in power series in small parameter.

Idea: Pose the solution as a power series in ε!

Ansatz:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

Stick into the polynomial:

$$x^2 + \varepsilon x - 1 = 0$$

$$\underbrace{(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)^2}_{x^2} + \varepsilon \underbrace{(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)}_{x} - 1 = 0$$

This is an eqn. for $x_0, x_1, \ldots$

↑ one

Idea: collect powers of ε!

$$x_0^2 \cancel{+ 2\varepsilon x_0 x_1 + 2\varepsilon^2 x_2 x_0} + \varepsilon^2 x_1^2 + \ldots$$

first term:

$$(x_0+\varepsilon x_1+\varepsilon^2 x_2+\dots)(x_0+\varepsilon x_1+\varepsilon^2 x_2+\dots)$$

$$= x_0^2+2\varepsilon x_0x_1+2\varepsilon^2 x_0x_2+\varepsilon^2 x_1^2+\dots$$

full eqn:

$$\underbrace{x_0^2+2\varepsilon x_0x_1+2\varepsilon^2 x_0x_2+\varepsilon^2 x_1^2+\dots}_{x^2}$$

$$+\underbrace{\varepsilon x_0+\varepsilon^2 x_1+\varepsilon^3 x_2+\dots}_{\varepsilon x}-1=0$$

collect powers of ε:

$$(x_0^2-1)\,\varepsilon^0+$$

$$(2x_0x_1+x_0)\,\varepsilon^1+$$

$$(2x_0x_2+x_1^2+x_1)\,\varepsilon^2+\dots=0$$

Set coeffns of powers ε to zero:

$\underline{\varepsilon^0:}$ $x_0^2 - 1 = 0 \Rightarrow \underline{\underline{x_0 = \pm 1}}$

$\underline{\varepsilon^1:}$ $2x_0 x_1 + x_0 = 0 \Rightarrow \underline{\underline{x_1 = -\frac{1}{2}}}$

$\underline{\varepsilon^2:}$ $2x_0 x_2 + x_1^2 + x_1 = 0$

$\Rightarrow x_2 = \pm \frac{1}{8}$

<u>etc.</u>

So our approximation/ansatz yields:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

$$= \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \dots$$

Identical to Taylor expansion but did not require knowledge of the exact solution.

Note the structure of the eqns:

- Lowest-order eqn (obtained from the lowest power of ε) is the full eqn. in the limit $\varepsilon = 0$. This eqn. was supposed to be simpler.
- Higher-order eqns provide corrections via a systematic hierarchy.
- Eqn. itself is satisfied to a specifiable accuracy (in powers of ε)

Looks very promising!

Issues: How do we "guess" the right form of the expansion?