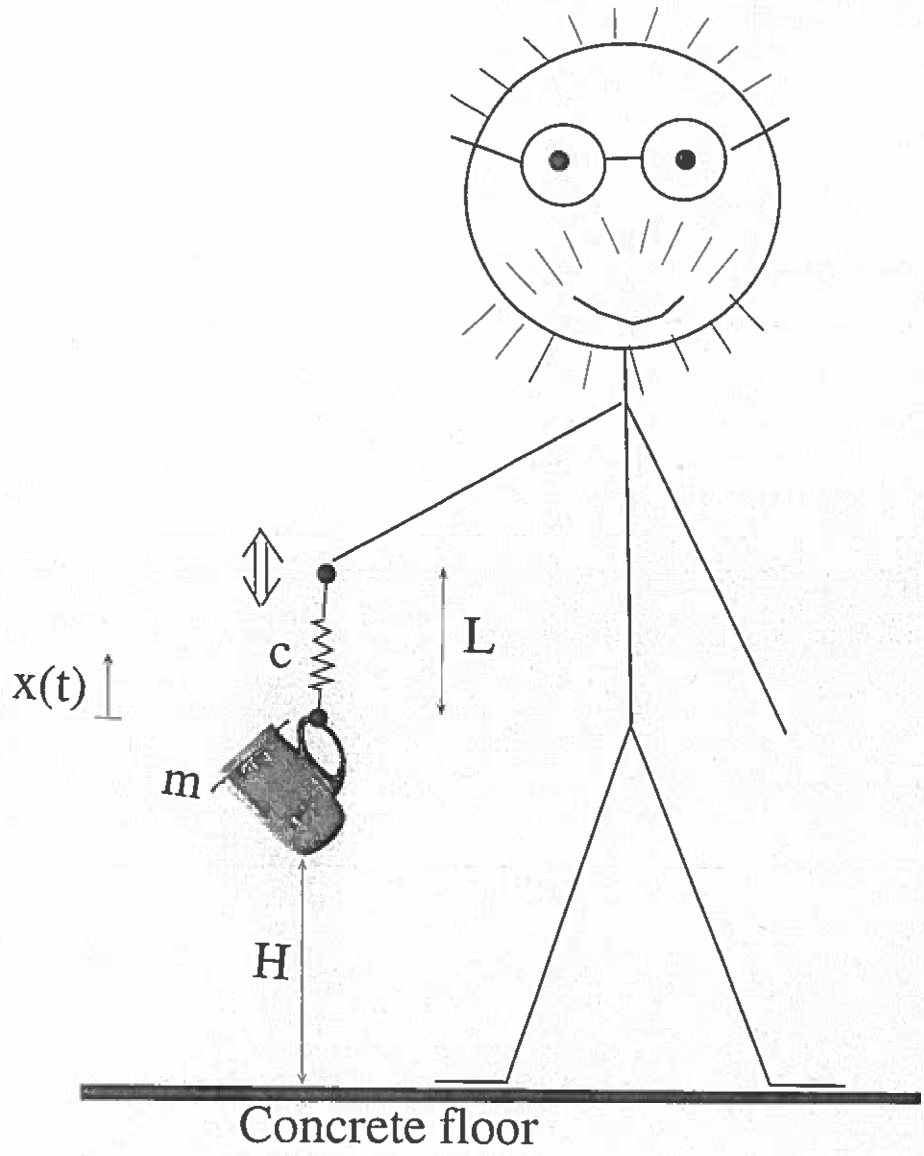
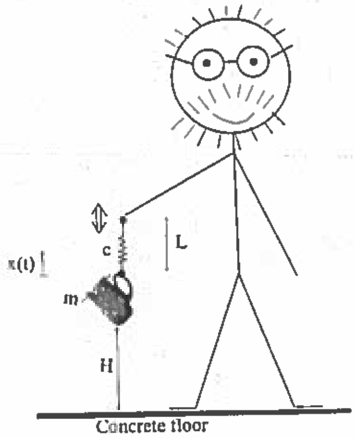
The experiment

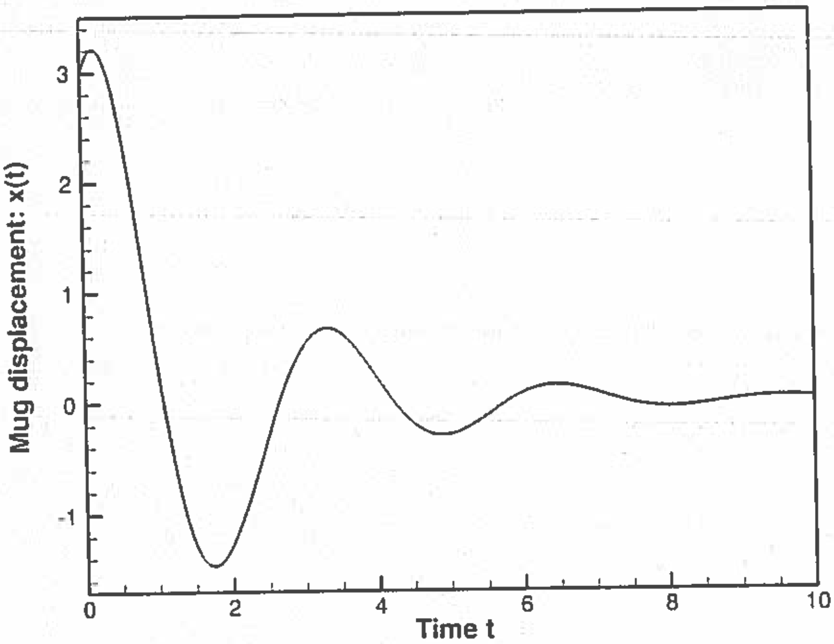


Experiment 1: Free oscillations



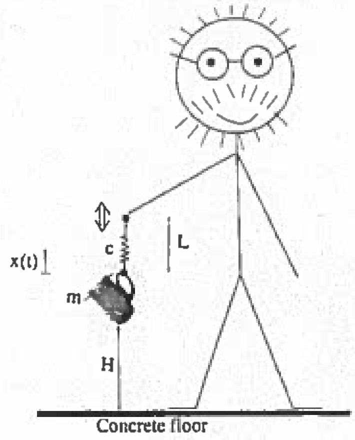
Procedure:

- Deflect mug from its rest position.
- "Let go" and observe the mug's time-dependent motion while keeping hand still.



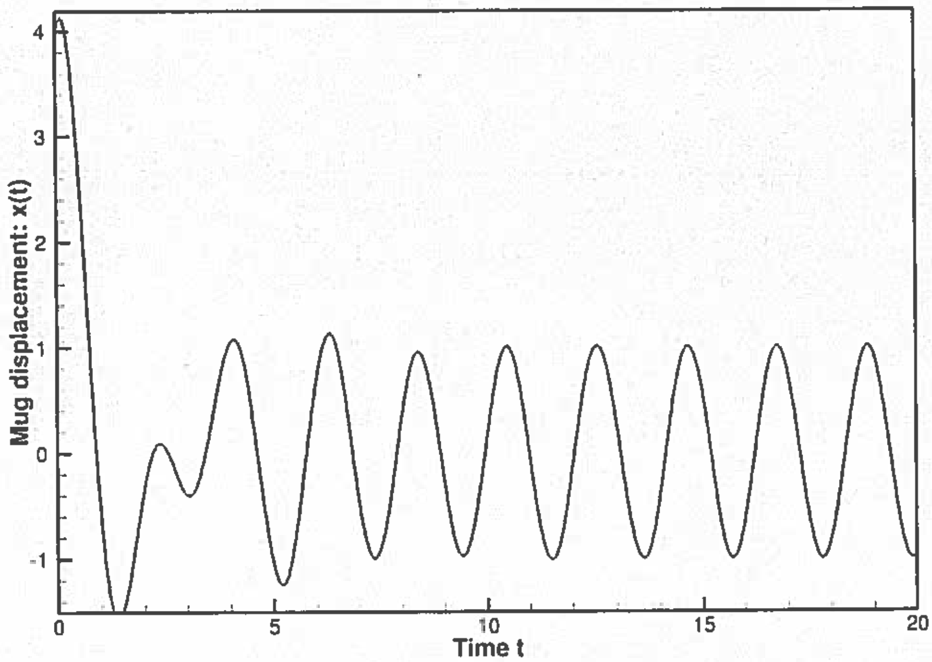
- Damped oscillation with certain characteristic frequency – the system's "eigen" frequency.

Experiment 2: Forced oscillations



Procedure:

- Start from rest.
- Perform time-harmonic oscillations with hand and observe the mug's time-dependent motion.



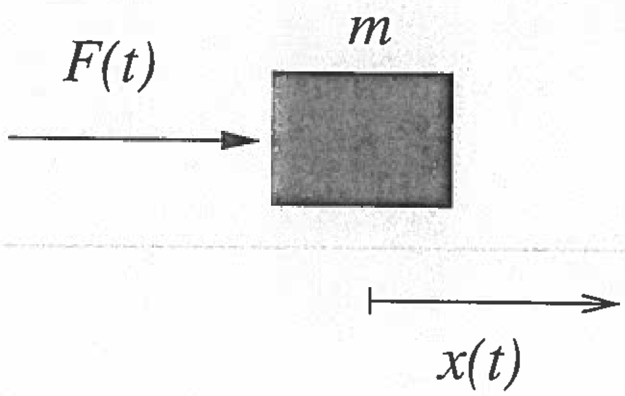
- "Initial transients", followed by time-harmonic "response" with same frequency as "forcing".
- Note: Amplitude of "response" depends on forcing frequency:
  - tiny for high frequencies,
  - same as forcing amplitude for very low frequencies,
  - very large for frequencies near the system's "eigen" frequency.

Everything you always wanted to know about mechanical oscillators but were afraid to ask

- The first half of MATH10222 is not directly concerned with mechanics.
- However, mechanical systems provide nice illustrations of many of the phenomena that we have discussed (or will discuss) in a more abstract mathematical setting.

I. Newton's law for one-dimensional motion

- In words: "*The sum of all forces acting on a particle of mass m is equal to its mass times its acceleration*"

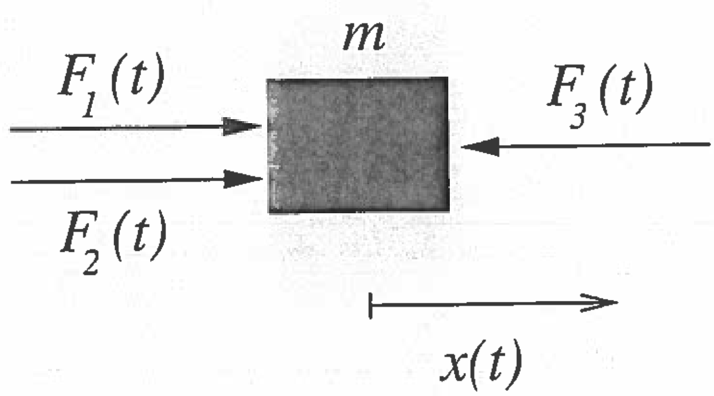


- Or, written as an equation:

$$m\frac{d^2x}{dt^2} = F(t)$$

I. Newton's law for one-dimensional motion (cont.)

- Here's an example with multiple forces



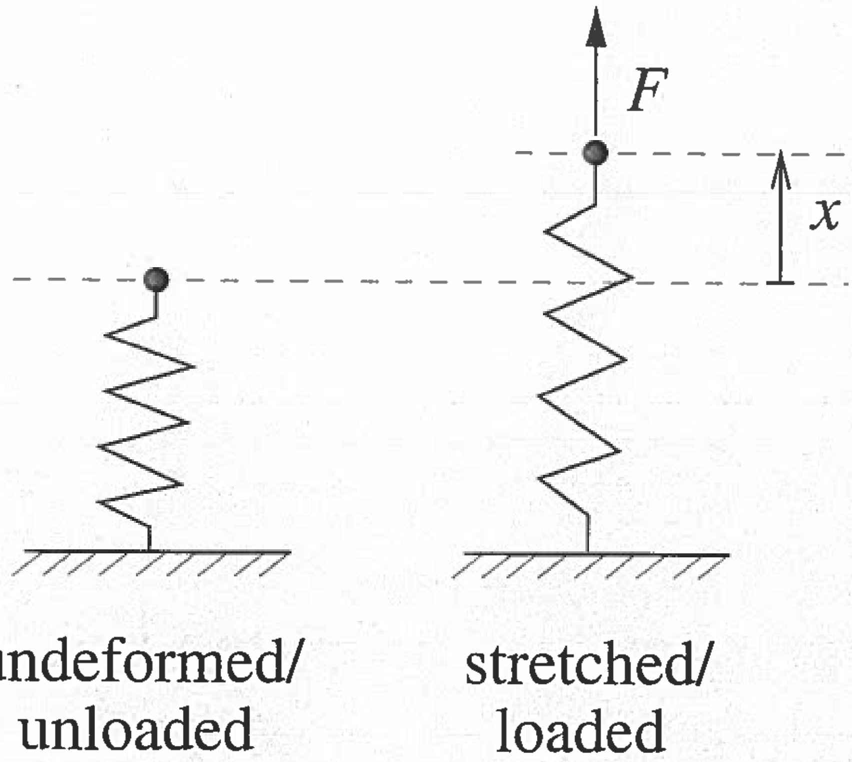
- In this case Newton's law becomes:

$$m\frac{d^2x}{dt^2} = F_1(t) + F_2(t) - F_3(t)$$

- Note the direction of the forces!

II. (Linearly) elastic springs

- Observation: When a spring is loaded by a force, F, its length increases by a certain amount, x, say.



- For a linearly elastic spring we have

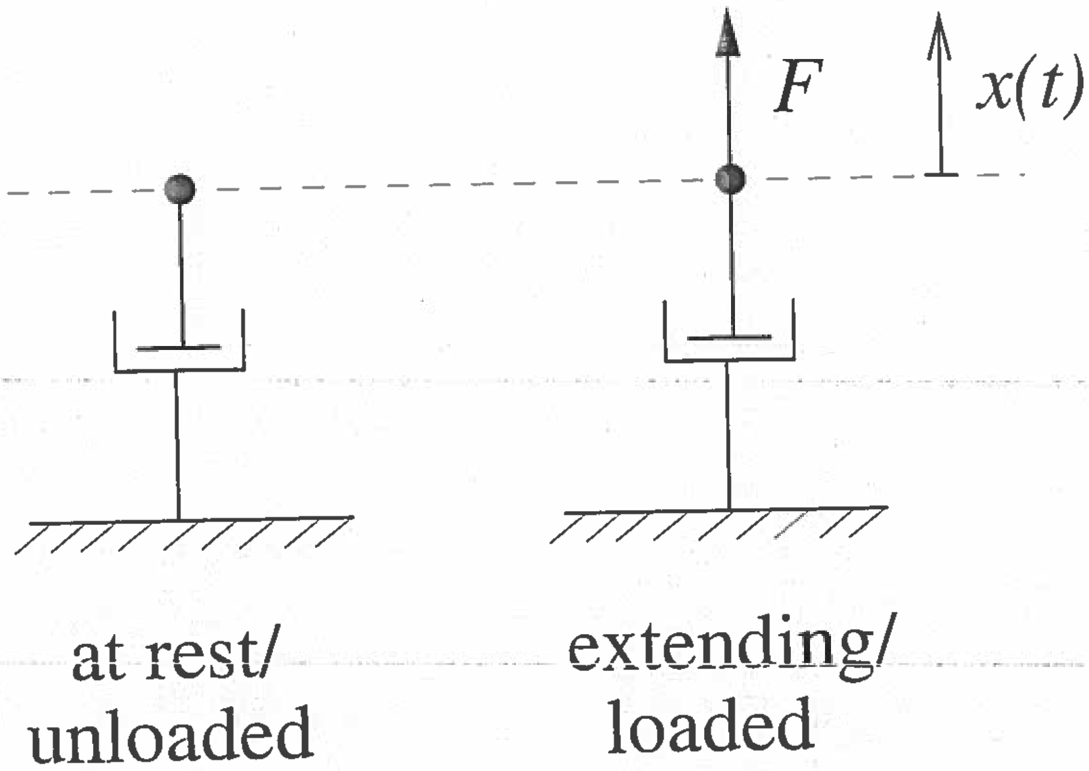
$$F = c\,x$$

where c is the "spring constant", a measure of its stiffness.

- Thus c indicates how strongly the spring resists its *static* extension.

III. (Linear) dampers

- Observation: When a damper is loaded by a force F its length increases at a rate dx/dt:



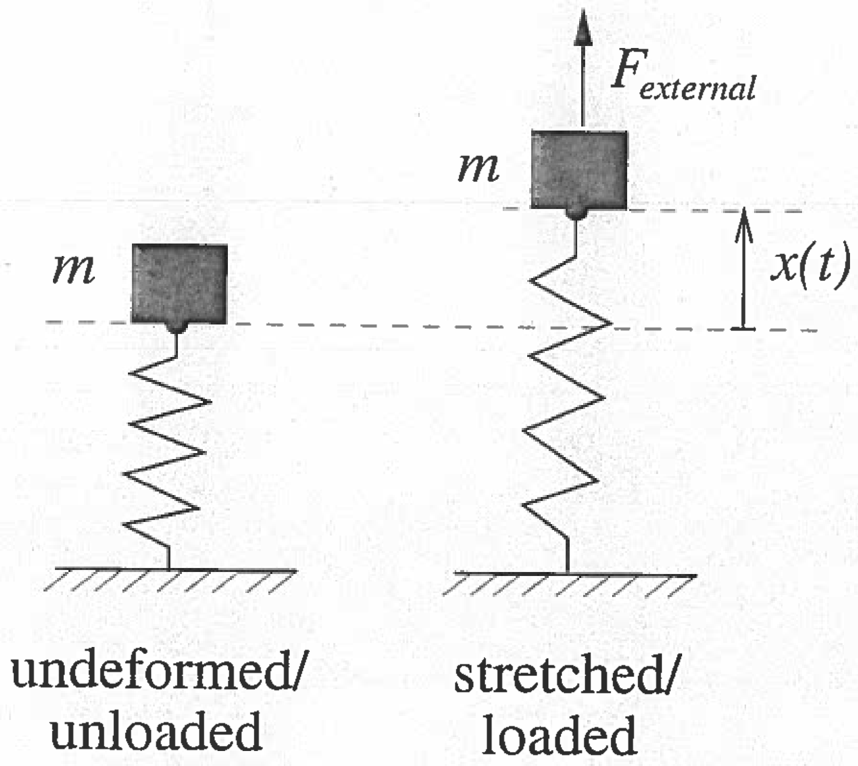
- For a linear damper we have

$$F = k\,\frac{dx}{dt}$$

where k is the "damping constant", a measure of how strongly the damper resists its *dynamic* extension.

IV. Putting it all together: "Action = Reaction"

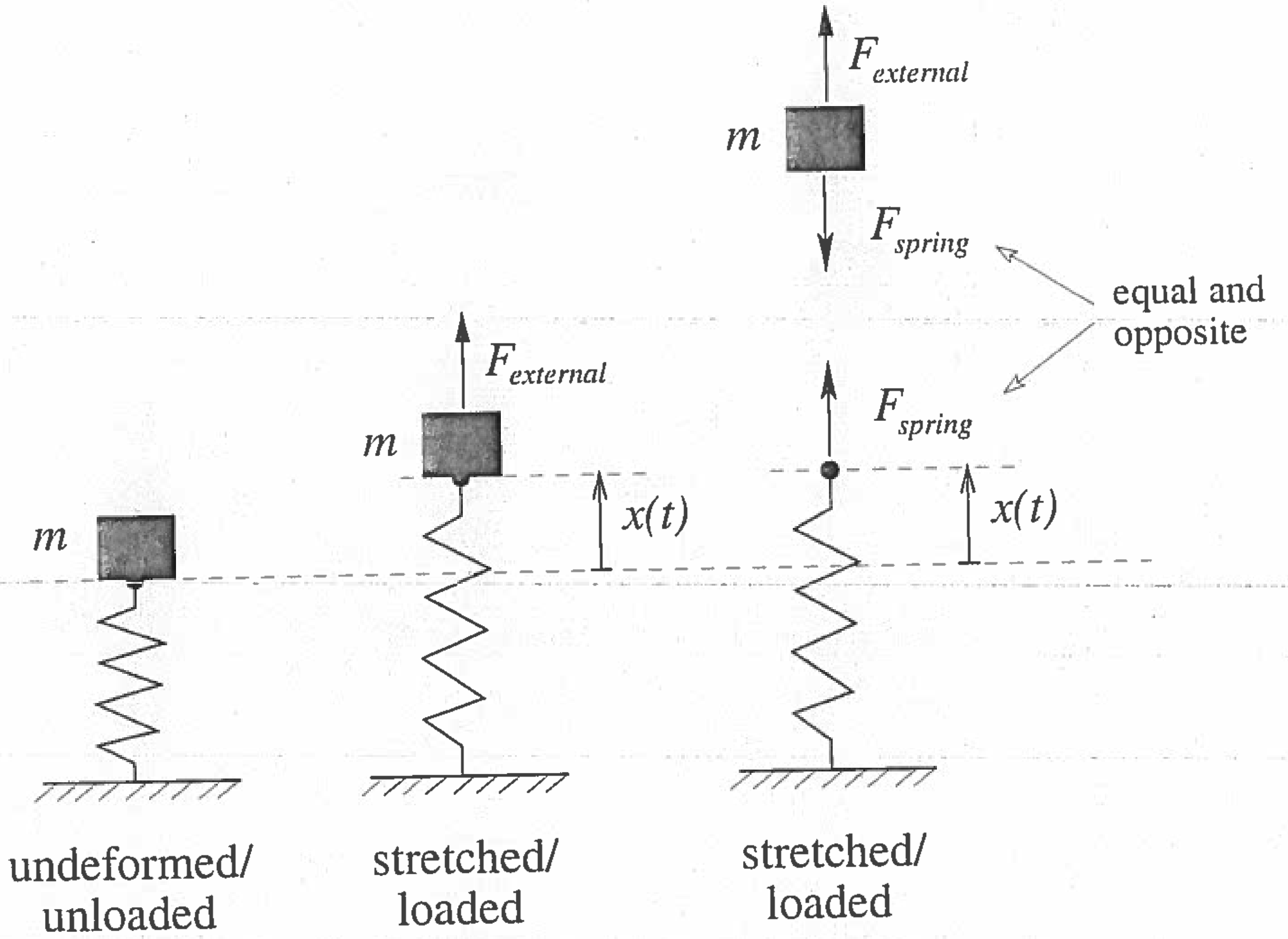
- Here is a mass m, attached to a spring of stiffness c, and loaded by a force, $F_{external}$.



- What is the equation of motion for the mass?
- Write down Newton's law for the mass.
- $\Longrightarrow$ What forces act on the mass?

IV. Putting it all together (cont.)

- "Action = Reaction": The spring pulls the mass and mass pulls the spring (in the opposite direction, obviously!):



- Thus Newton's law states

$$m\frac{d^2x}{dt^2} = F_{external} - F_{spring},$$

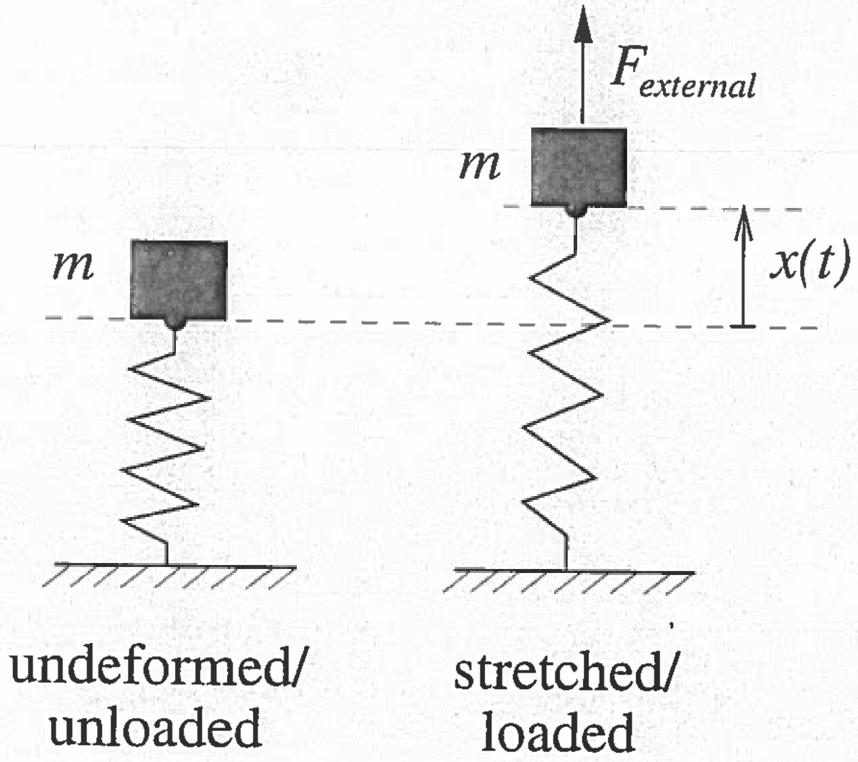
or, using what we've just learned about linear springs:

$$m\frac{d^2x}{dt^2} = F_{external} - cx.$$

IV. Putting it all together (cont.)

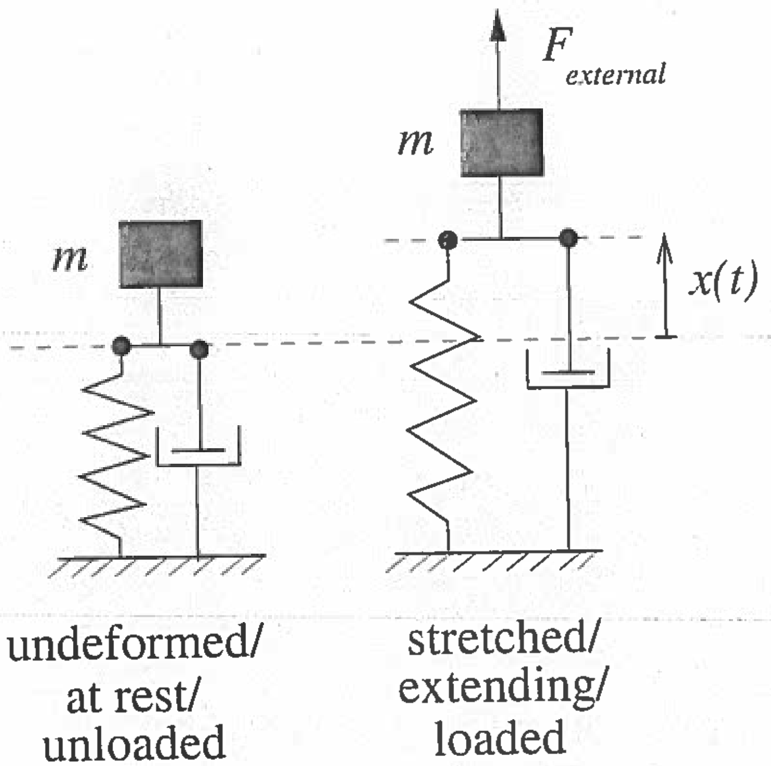
- Rewrite to the standard form of a second-order ODE for $x(t)$:

$$m\frac{d^2x}{dt^2} + cx = F_{external}.$$



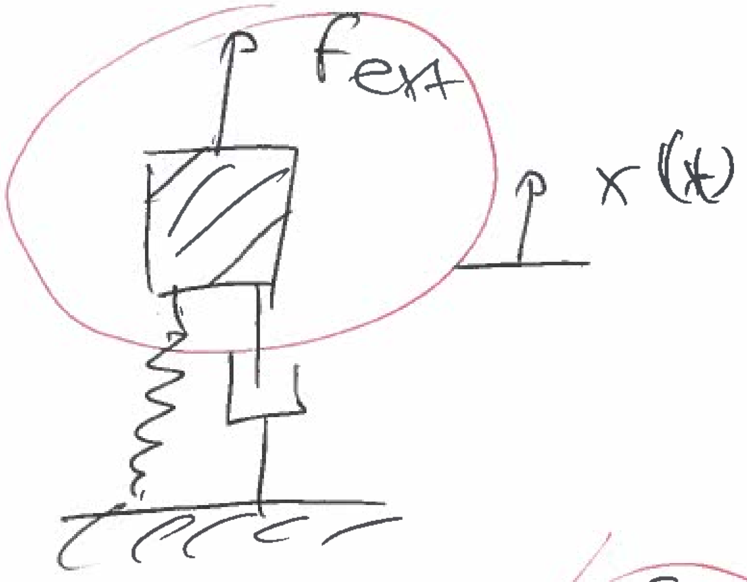
Exercise: Try it for yourself

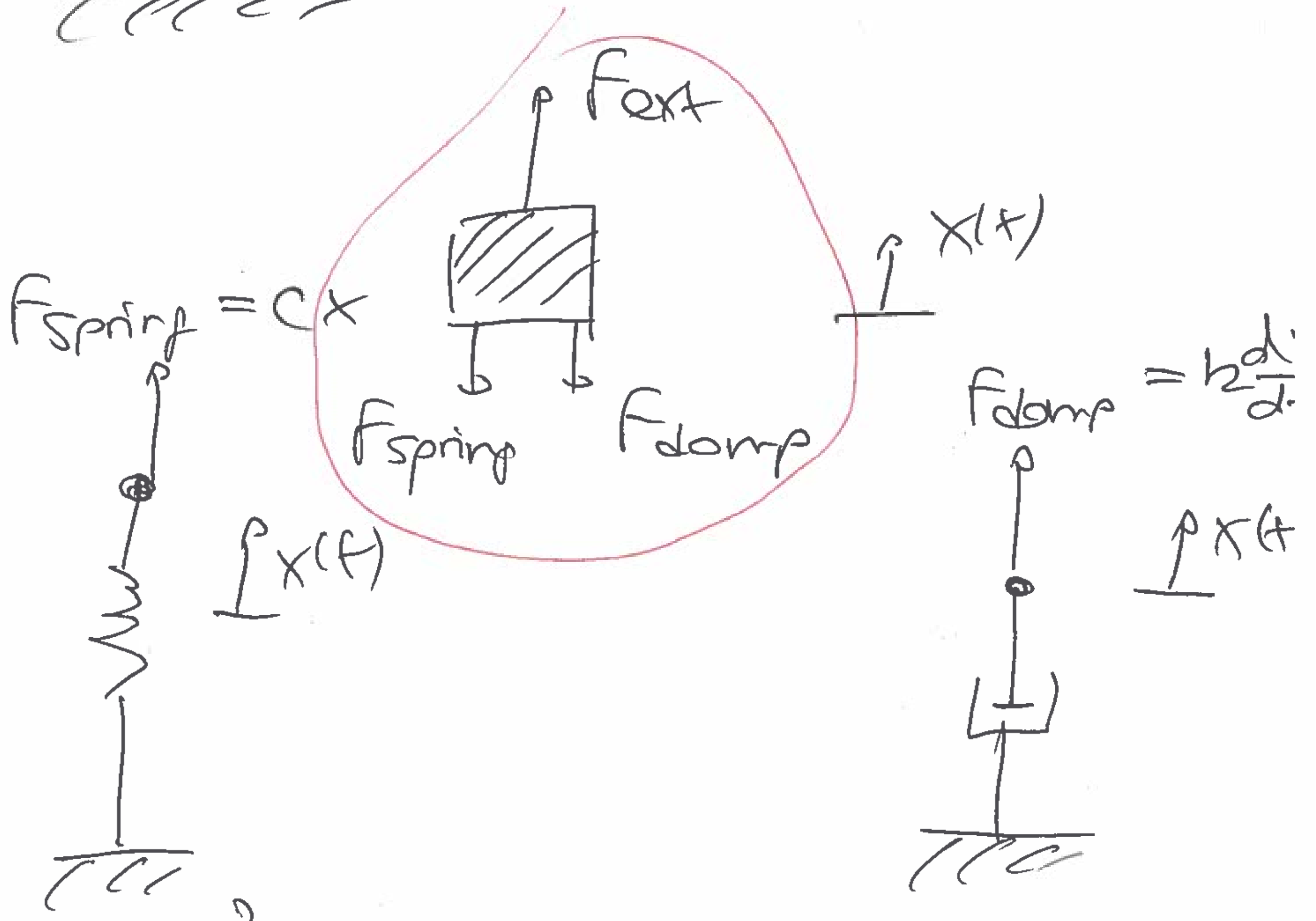
- Here is a mass m, attached to a spring of stiffness c, and a damper (damping constant k), loaded by a force $F_{external}$.



- Show that the equation of motion for the mass is

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = F_{external}$$





$$m\frac{d^2x}{dt^2} = f_{ext} - f_{spring} - f_{damp}$$

$$m\frac{d^2x}{dt^2} = f_{ext} - cx - k\frac{dx}{dt}$$

$$\boxed{m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = f_{ext}(t)}$$

2^{nd} order linear const. coeffn. ODE. for $x(t)$.

Rewrite as:

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = f(t)$$

where:

$$\delta = \frac{b}{2m} \geq 0$$

$$\omega^2 = \frac{c}{m} \geq 0$$

$$f(t) = \frac{F_{ext}(t)}{m}$$

IC: $x(t=0) = X_0$ initial posn

$\left.\frac{dx}{dt}\right|_{t=0} = V_0$ initial veloc.

3

The unforced case, $f(t)=0$

$$\ddot{x}+2\delta\dot{x}+\omega^2 x=0$$

char. poly:

$$\lambda^2+2\delta\lambda+\omega^2=0$$

$$\lambda_{12}=-\delta\pm\sqrt{\delta^2-\omega^2}$$

Four cases:

① Purely damped motion: $\delta>\omega$

(strong damping)

$\Rightarrow$ two distinct real roots

$$x(t)=Ae^{\left(-\delta+\sqrt{\delta^2-\omega^2}\right)t}+Be^{\left(-\delta-\sqrt{\delta^2-\omega^2}\right)t}$$

Note: both values of λ are negative $\Rightarrow \lim\limits_{t\to\infty} x(t)=0$ without any oscillation.

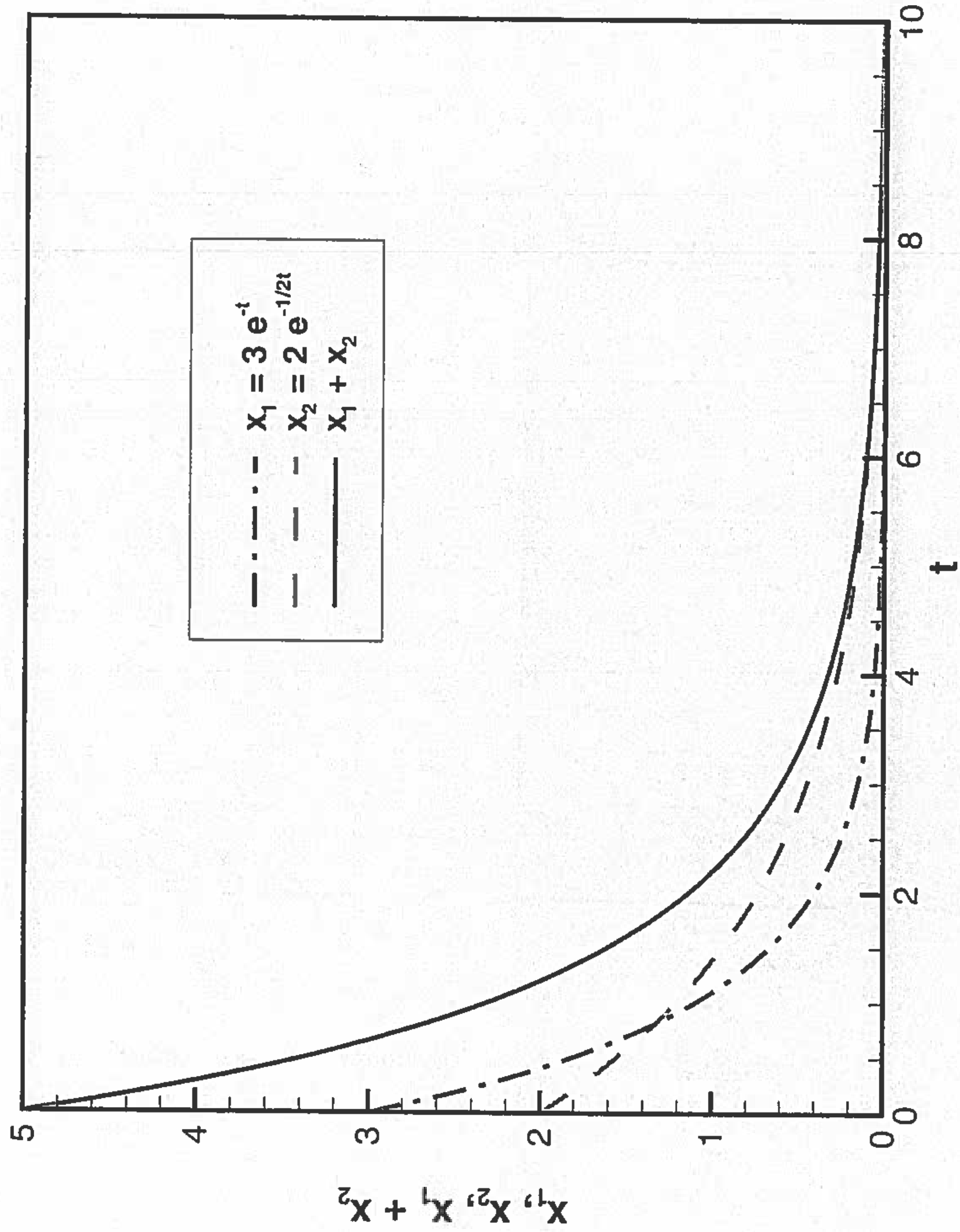


Figure 1: Illustration of a purely damped motion. The mass approaches its equilibrium position $x = 0$ monotonically.

② Critically damped motion ; $\delta = \omega$

repeated roots $\lambda_{12} = -\delta$

$$x(t) = A e^{-\delta t} + B t e^{-\delta t}$$

Note: $\delta > 0 \Rightarrow$

$\lim_{t \to \infty} x(t) = 0$ with at most one crossing of $x = 0$, depending on IC.

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③ Damped oscillation ; $\delta < \omega$

$$\lambda_{12} = -\delta \pm i\sqrt{\omega^2 - \delta^2}$$

$$x(t) = e^{-\delta t}\left(A \cos\left(\sqrt{\omega^2-\delta^2}\, t\right) + B \sin\left(\sqrt{\omega^2-\delta^2}\, t\right)\right)$$

Damped oscillation with frequency $\sqrt{\omega^2 - \delta^2}$. Note $\delta = \frac{k}{2m}$

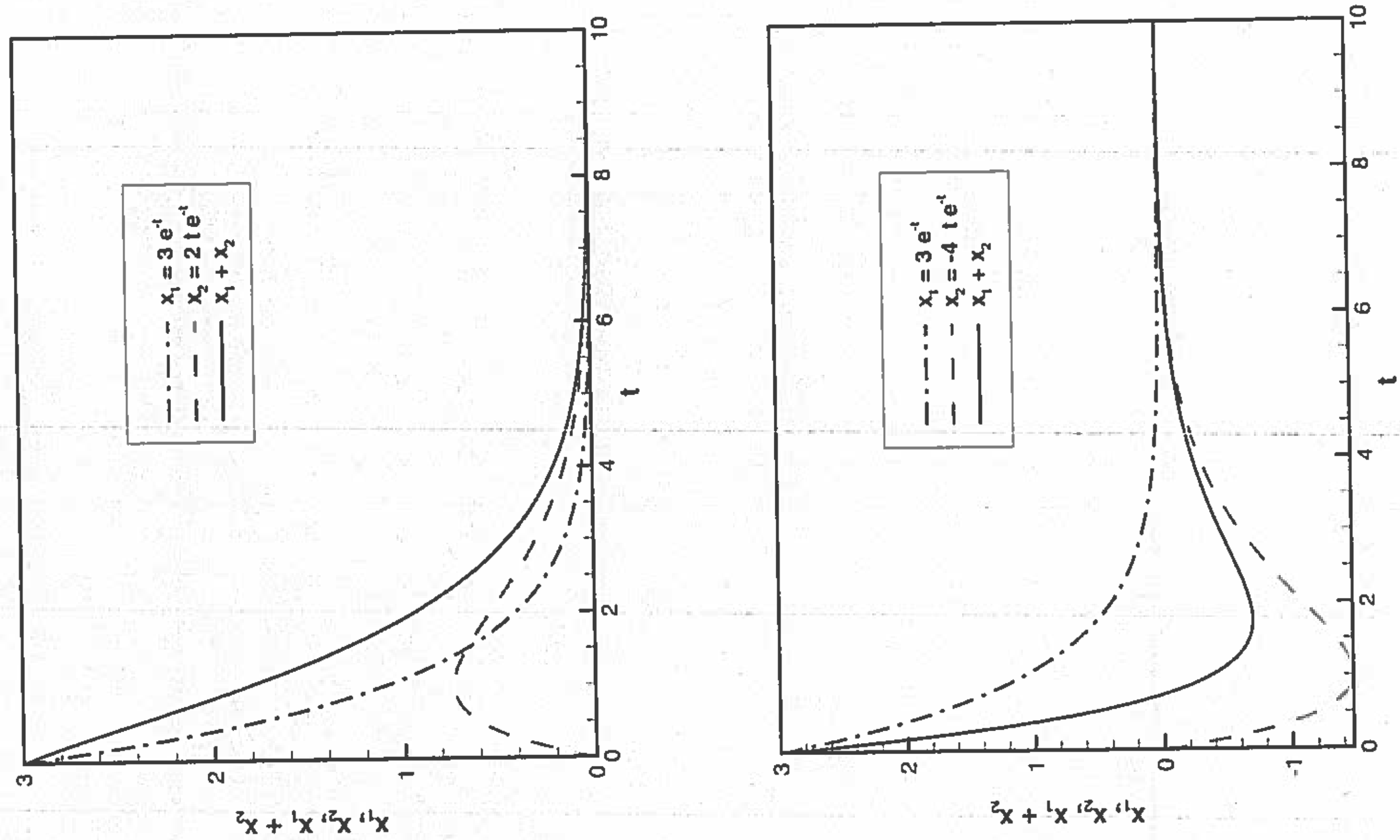


Figure 2: Illustration of critically damped motions. The mass approaches its equilibrium position, $x = 0$, with at most one "overshoot".

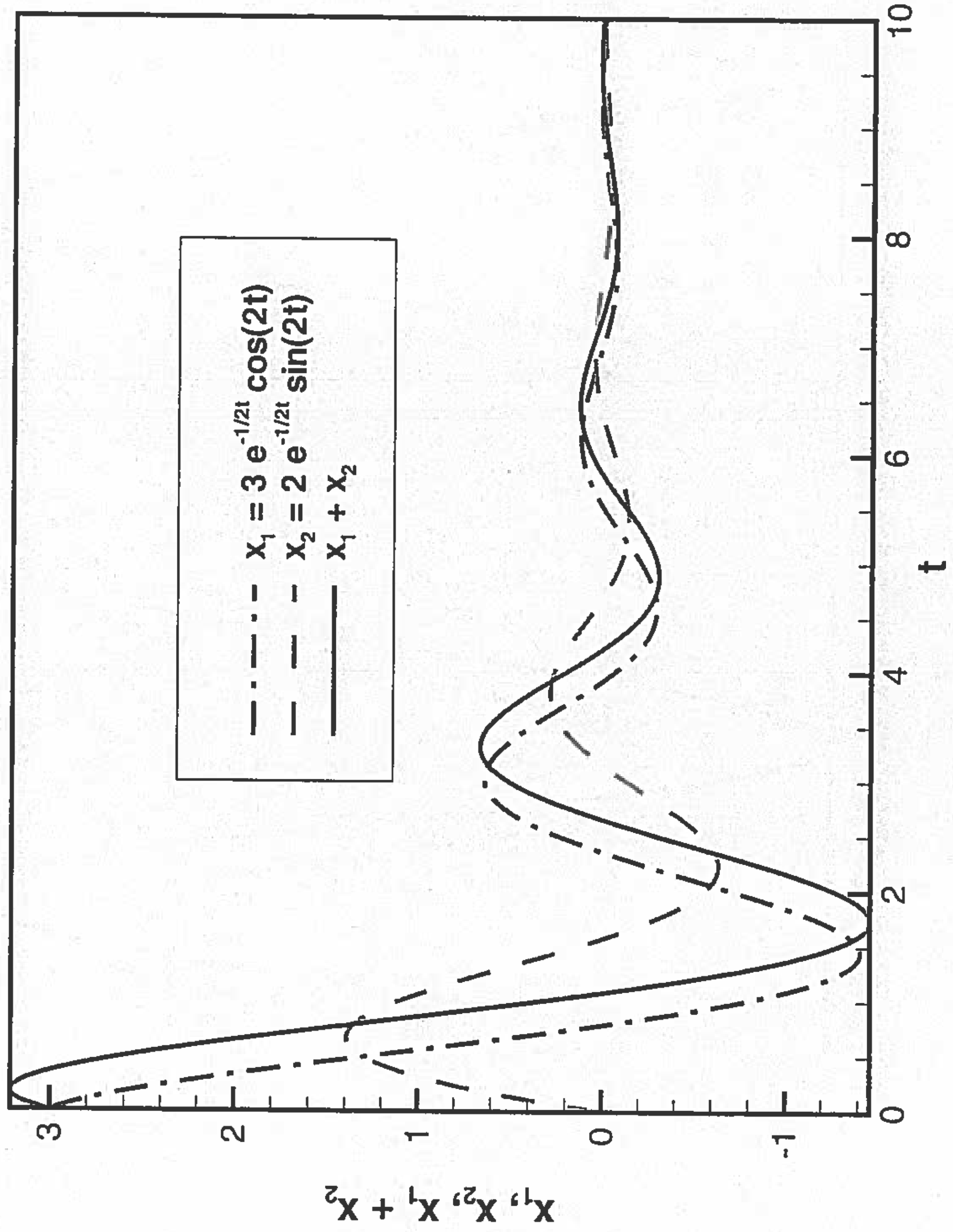


Figure 3: Illustration of a damped oscillation. The mass oscillates about its equilibrium position $x = 0$ and the amplitude of the oscillations decays exponentially.

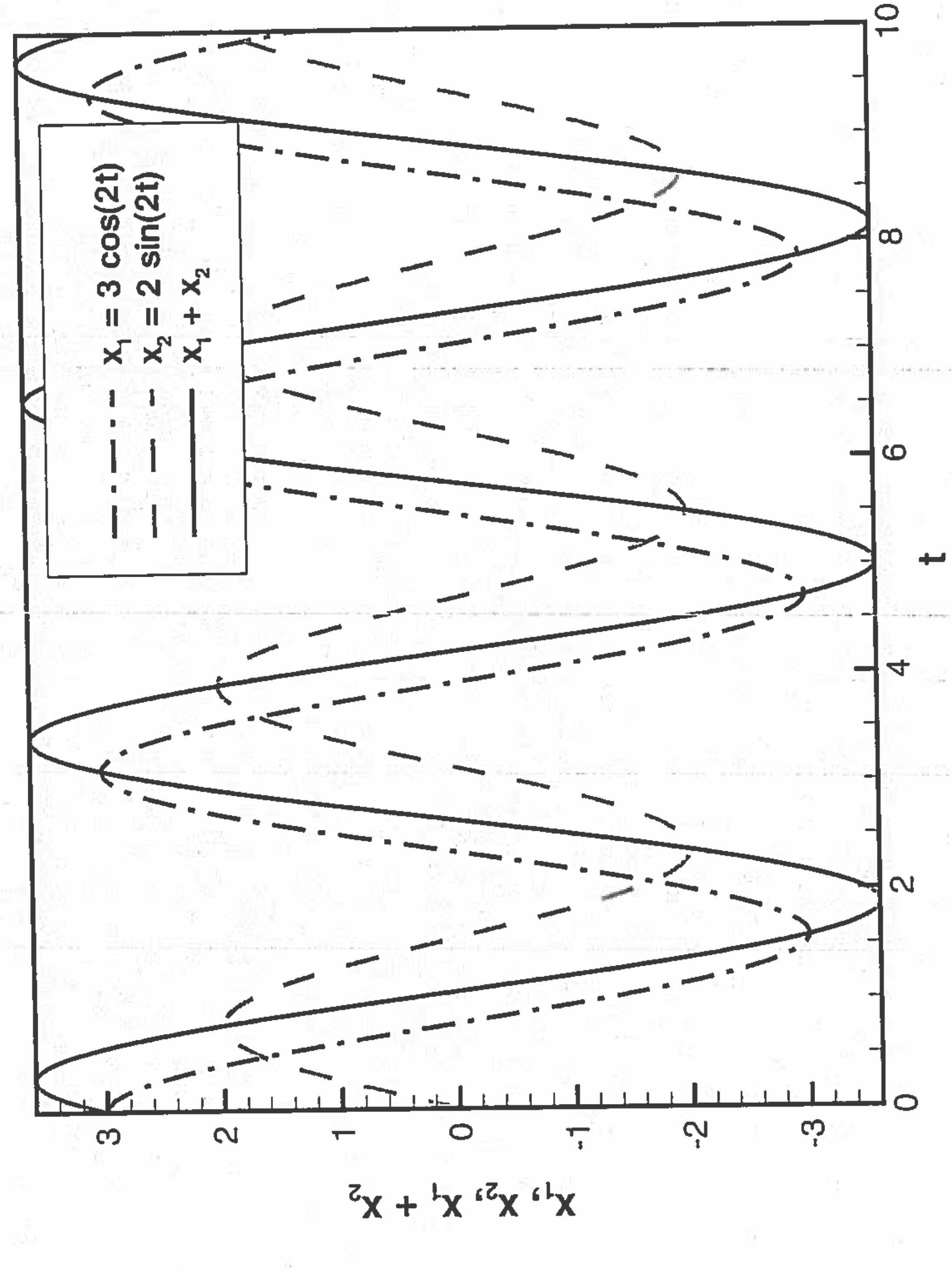


Figure 4: Illustration of an undamped oscillation. The mass performs harmonic oscillations about its equilibrium position $x = 0$.

indicates the rate of exponential decay of the oscillation. (5

(4) undamped oscillation $\delta = 0$

$$\lambda_{12} = \pm i\omega$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

undamped oscillation with the system's eifen frequency $\omega = \sqrt{\frac{c}{m}}$.

Forced oscillation:

Periodic forcing & resonance

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = f(t)$$

$$f(t) = \hat{f}\sin(\Omega t) \text{ or } \hat{f}\cos(\Omega t)$$

L6

Can do both cases at the same time by considering

$$f(t) = \hat{f} e^{i\Omega t}$$

& then extract real or imag. part of the soln for cos or sin forcing respectively.

Ansatz:

given!

$$x_p = X e^{i\Omega t}$$

$$\dot{x}_p = i\Omega X e^{i\Omega t}$$

$$\ddot{x}_p = -\Omega^2 X e^{i\Omega t}$$

into ODE:

$$\Rightarrow X \cancel{e^{i\Omega t}} \Big(\underbrace{-\Omega^2}_{\ddot{x}} + \underbrace{2\delta i \Omega}_{\dot{x}} + \underbrace{\omega^2}_{x} \Big) = \hat{f} \cancel{e^{i\Omega t}}$$

$$X = \frac{\hat{f}}{(\omega^2 - \Omega^2) + i(2\delta\Omega)}$$ complex

$$\bar{X} = X_{real} + i X_{imag} = |\bar{X}| e^{i\varphi}$$

$$\varphi = \arg(\bar{X})$$

Note: $|\bar{X}|$ gives us the amplitude of the resulting harmonic motion.

$$|\bar{X}| = \frac{\hat{f}}{\sqrt{(\omega^2 - \Omega^2)^2 + (2\delta\Omega)^2}}$$

$$|\bar{X}| = \frac{\hat{f}/\omega^2}{\sqrt{\left(1 - \left(\frac{\Omega}{\omega}\right)^2\right)^2 + \left(2\left(\frac{\delta}{\omega}\right)\left(\frac{\Omega}{\omega}\right)\right)^2}}$$

ratio of the forcing freq Ω to eigenfreq. ω

ratio of damping constant to spring stiffness

Where does $\hat{f}/\omega^2$ come from / what does it mean?

Recall:

$$m\ddot{x} + k\dot{x} + cx = \hat{F}_{ext}\cos(\Omega t)$$

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = \hat{f}\cos(\Omega t)$$

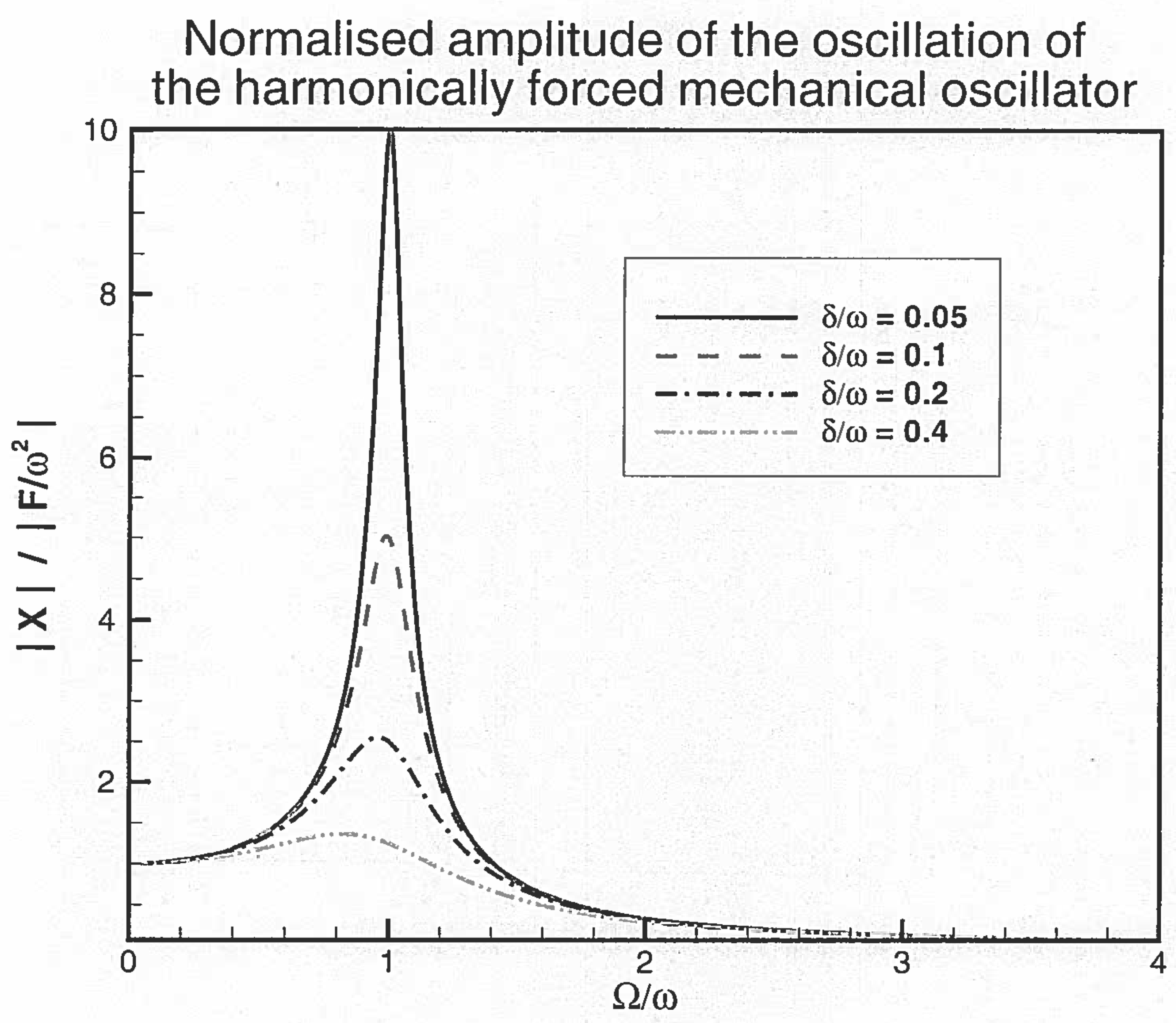
This implies: $\hat{f} = \dfrac{\hat{F}_{ext}}{m}$

$\omega^2 = \dfrac{c}{m}$

$$\Rightarrow \frac{\hat{f}}{\omega^2} = \frac{\hat{F}_{ext}}{c}$$

$\hat{F}_{ext}$ — max. force exerted onto spring of stiffness c

c — Static extension of the spring when subjected to F_{ext}.



Normalised amplitude of the oscillation of
the harmonically forced mechanical oscillator
| X | / | F/ω² |
Ω/ω
δ/ω = 0.05
δ/ω = 0.1
δ/ω = 0.2
δ/ω = 0.4
0
1
2
3
4
2
4
6
8
10