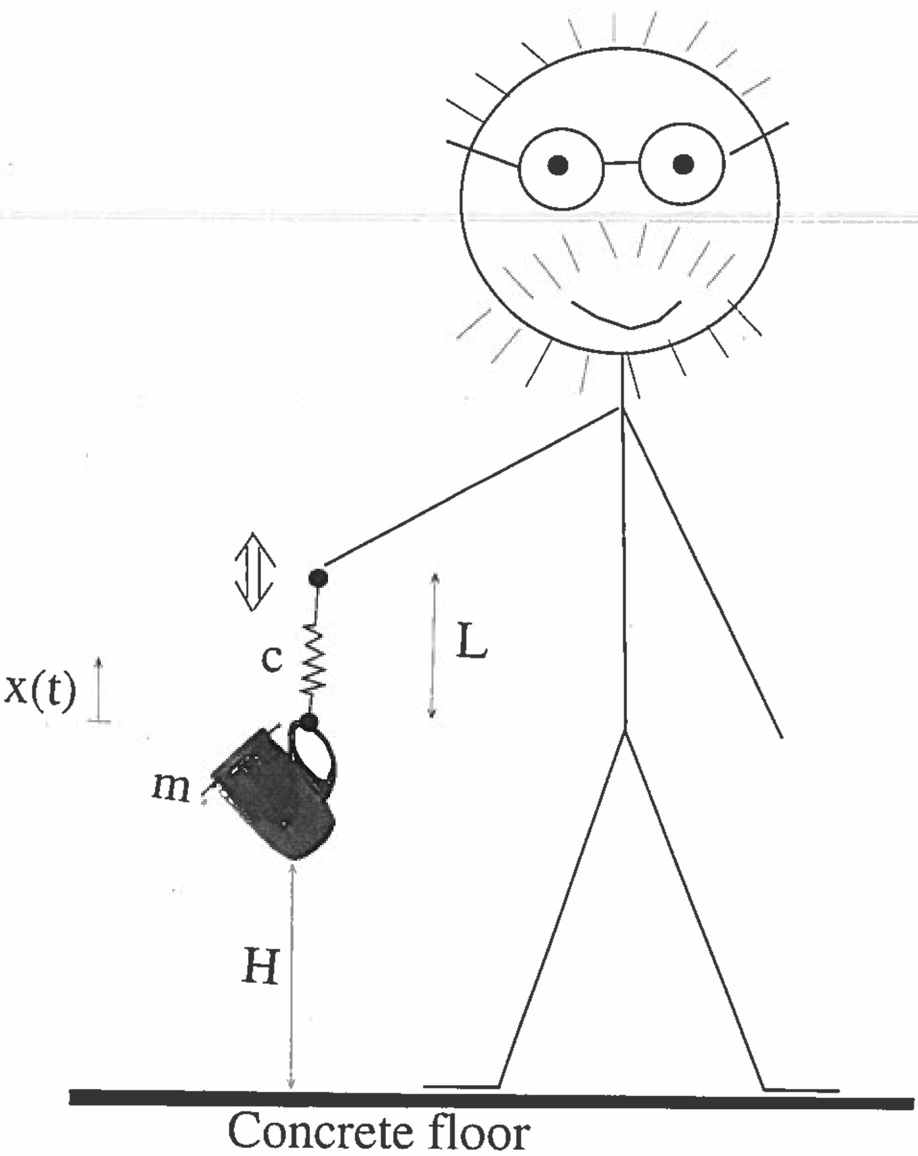
The experiment



Experiment 1: Free oscillations

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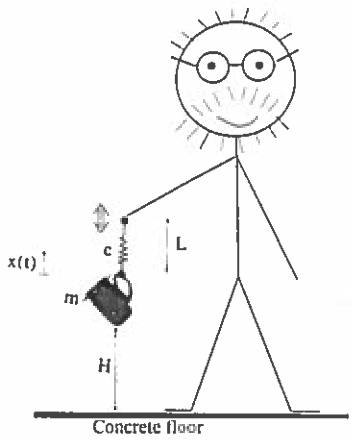
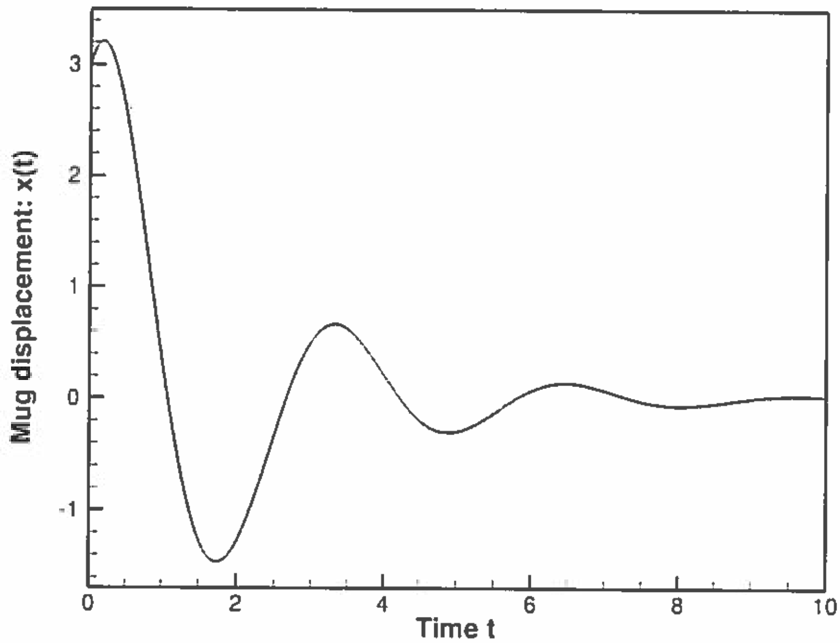


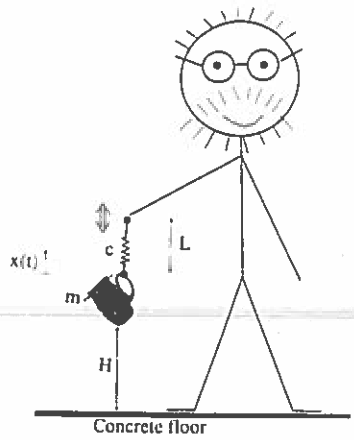
Procedure:

- Deflect mug from its rest position.
- "Let go" and observe the mug's time-dependent motion while keeping hand still.



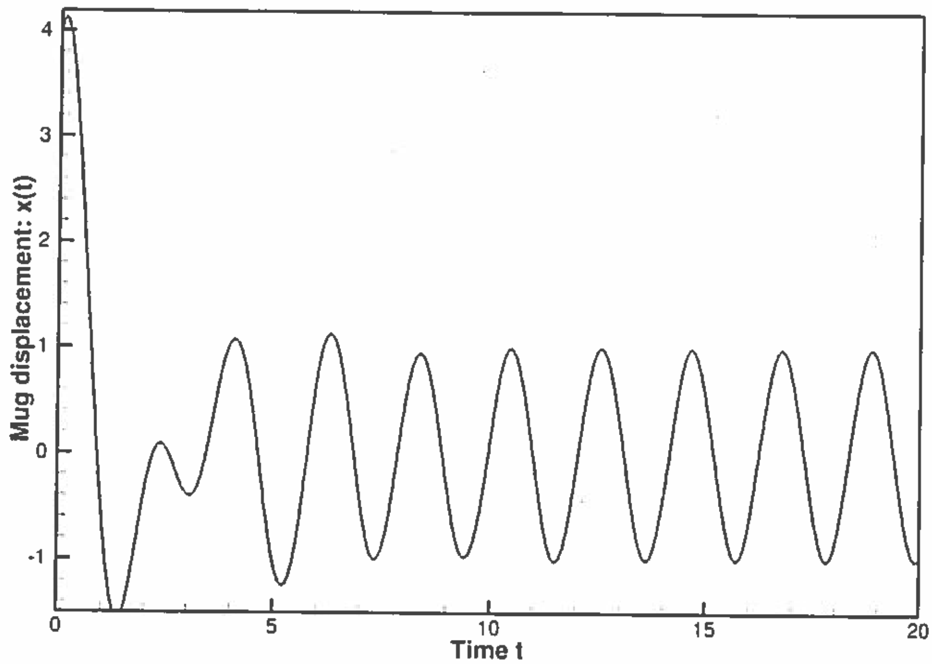
- Damped oscillation with certain characteristic frequency – the system's "eigen"frequency.

Experiment 2: Forced oscillations



Procedure:

- Start from rest.
- Perform time-harmonic oscillations with hand and observe the mug's time-dependent motion.



- "Initial transients", followed by time-harmonic "response" with same frequency as "forcing".
- Note: Amplitude of "response" depends on forcing frequency:
  - tiny for high frequencies,
  - same as forcing amplitude for very low frequencies,
  - very large for frequencies near the system's "eigen" frequency.

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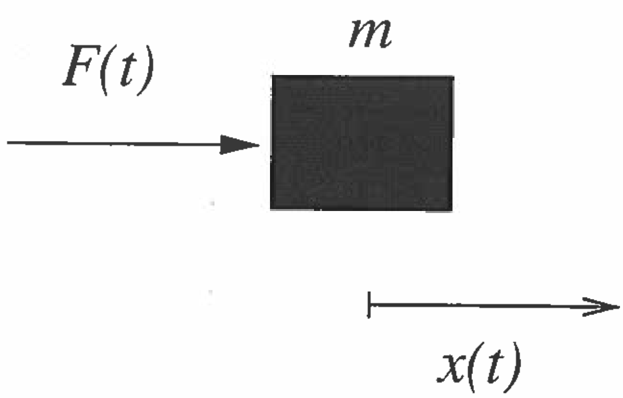
Everything you always wanted to know about mechanical oscillators but were afraid to ask

- The first half of MATH10222 is not directly concerned with mechanics.
- However, mechanical systems provide nice illustrations of many of the phenomena that we have discussed (or will discuss) in a more abstract mathematical setting.

I. Newton's law for one-dimensional motion

- In words: "*The sum of all forces acting on a particle of mass m is equal to its mass times its acceleration*"

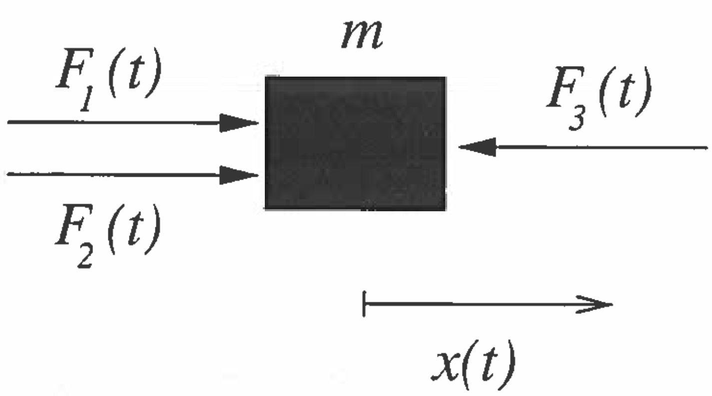


- Or, written as an equation:

$$m\frac{d^2x}{dt^2} = F(t)$$

I. Newton's law for one-dimensional motion (cont.)

- Here's an example with multiple forces



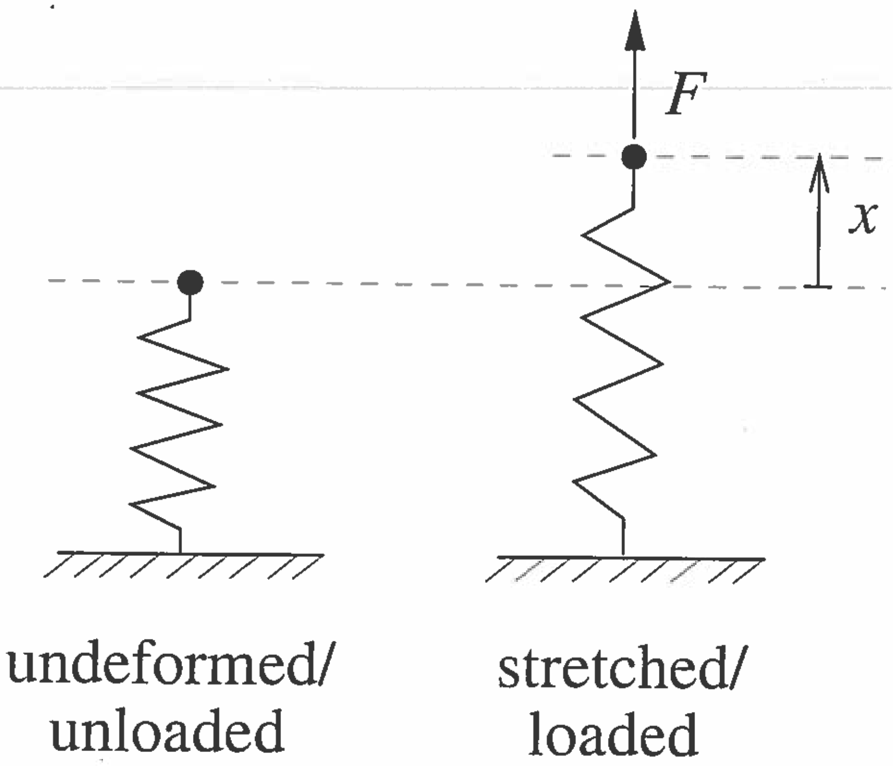
- In this case Newton's law becomes:

$$m\frac{d^2x}{dt^2} = F_1(t) + F_2(t) - F_3(t)$$

- Note the direction of the forces!

II. (Linearly) elastic springs

- Observation: When a spring is loaded by a force, F, its length increases by a certain amount, x, say.



- For a linearly elastic spring we have

$$F = c\,x$$

where c is the "spring constant", a measure of its stiffness.

- Thus c indicates how strongly the spring resists its *static* extension.

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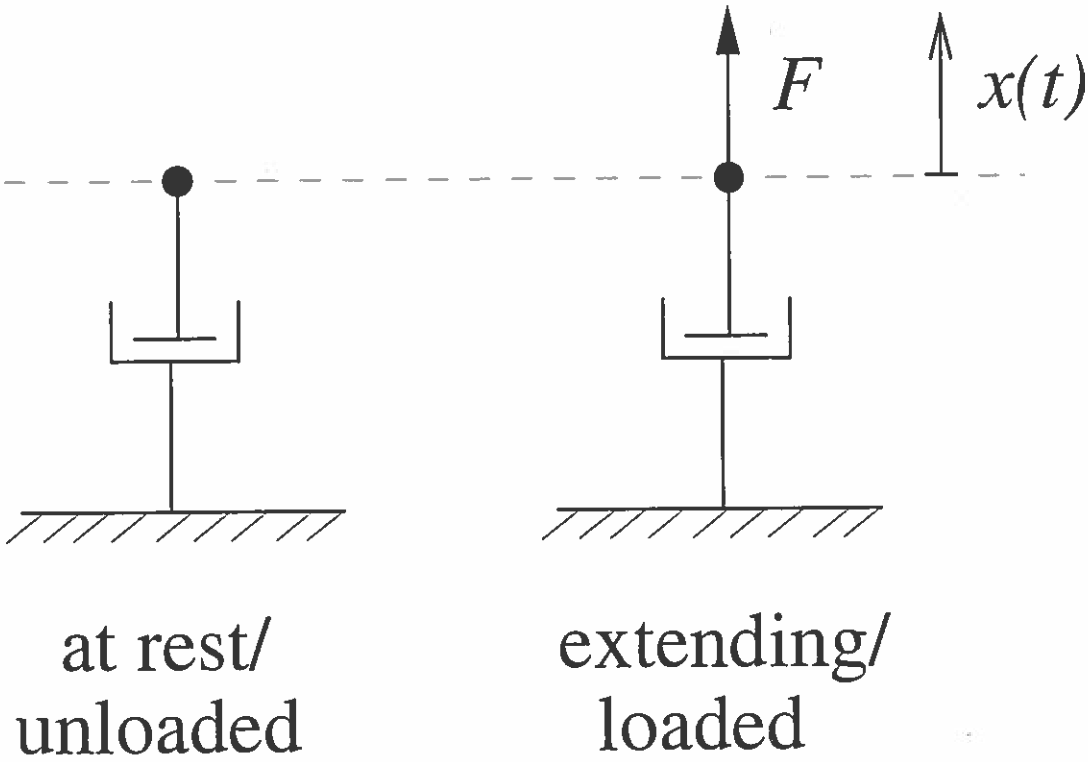
III. (Linear) dampers

- Observation: When a damper is loaded by a force F its length increases at a rate dx/dt:



- For a linear damper we have

$$F = k\,\frac{dx}{dt}$$

where k is the "damping constant", a measure of how strongly the damper resists its *dynamic* extension.

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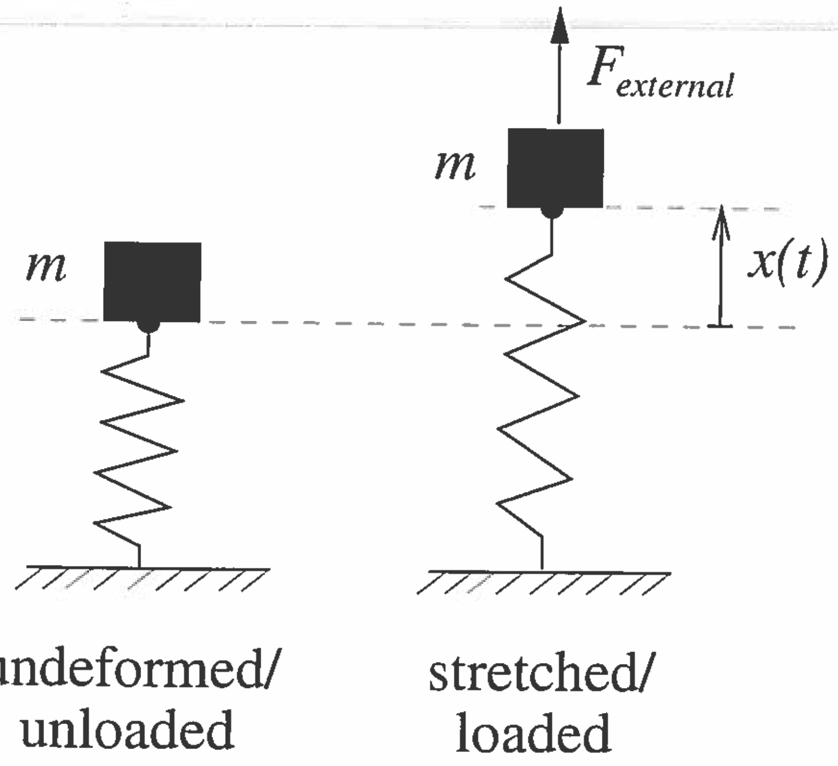
IV. Putting it all together: "Action = Reaction"

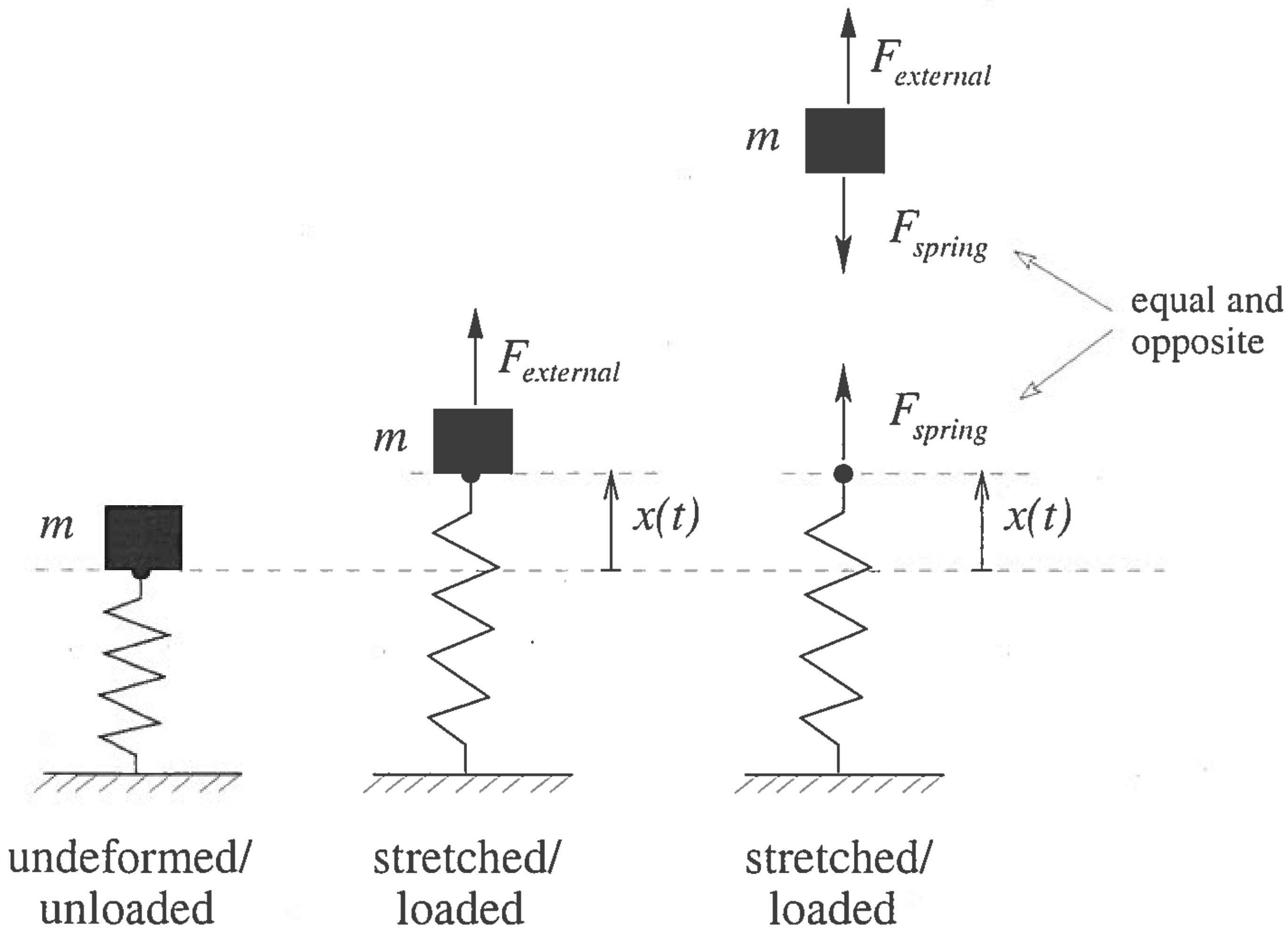
- Here is a mass m, attached to a spring of stiffness c, and loaded by a force, $F_{external}$.



- What is the equation of motion for the mass?
- Write down Newton's law for the mass.
- $\Longrightarrow$ What forces act on the mass?

IV. Putting it all together (cont.)

- "Action = Reaction": The spring pulls the mass and mass pulls the spring (in the opposite direction, obviously!):



- Thus Newton's law states

$$m\frac{d^2x}{dt^2} = F_{external} - F_{spring},$$

or, using what we've just learned about linear springs:

$$m\frac{d^2x}{dt^2} = F_{external} - cx.$$

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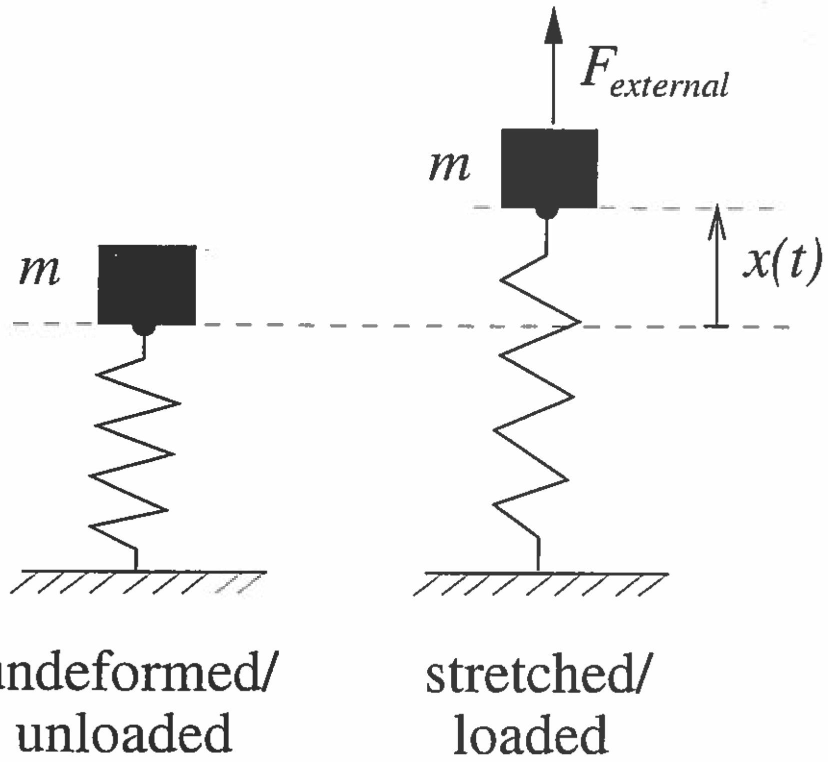
IV. Putting it all together (cont.)

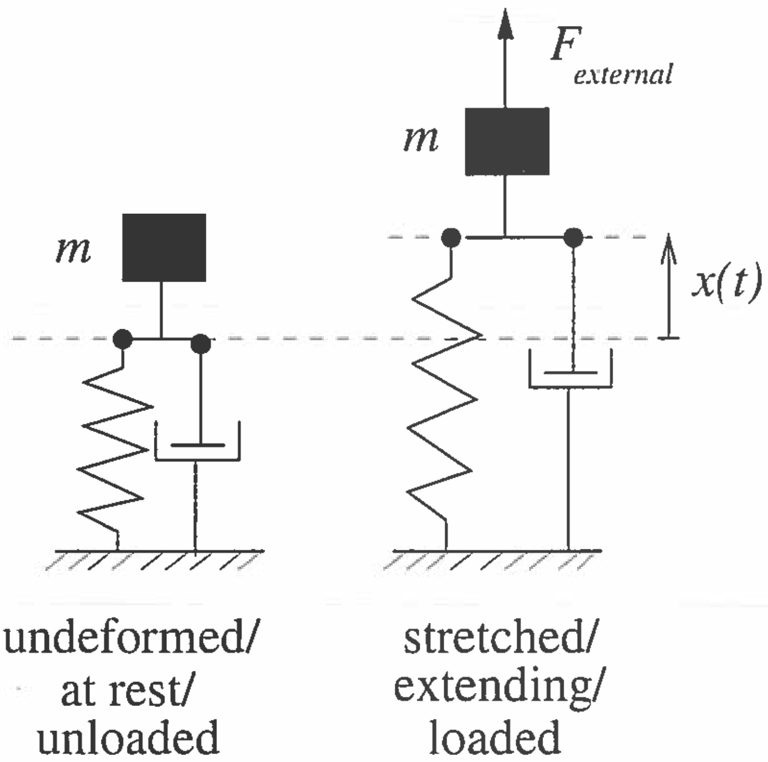
- Rewrite to the standard form of a second-order ODE for $x(t)$:

$$m\frac{d^2x}{dt^2} + cx = F_{external}.$$



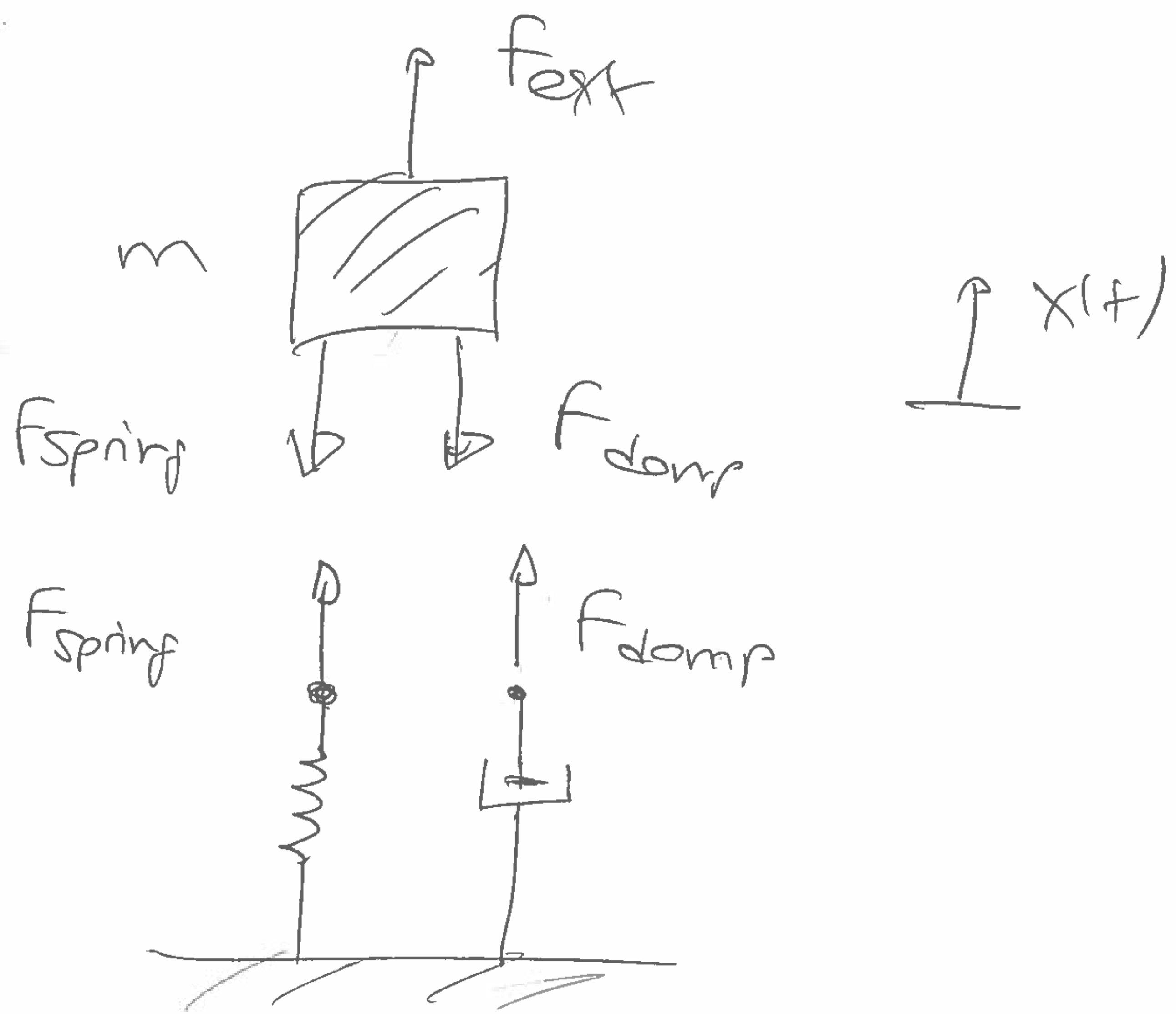
Exercise: Try it for yourself

- Here is a mass m, attached to a spring of stiffness c, and a damper (damping constant k), loaded by a force $F_{external}$.



- Show that the equation of motion for the mass is

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = F_{external}$$



$$m\frac{d^2x}{dt^2} = f_{ext} - \underbrace{f_{spring}}_{cx} - \underbrace{f_{damp}}_{k\frac{dx}{dt}}$$

$$\boxed{m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = f_{ext}(t)}$$

<u>IC:</u> $x(t=0) = x_0$ initial posn

$\left.\frac{dx}{dt}\right|_{t=0} = v_0$ initial veloc.

Rewrite:

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = f(t)$$

where:

$$\delta = \frac{k}{2m} > 0$$

$$\omega^2 = \frac{c}{m} > 0$$

$$f(t) = \frac{F_{ext}(t)}{m}$$

The unforced case: $f(t) = 0$

Described by the homog. ODE

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = 0$$

$$x \sim e^{\lambda t} \quad \ldots$$

$$\lambda_{12} = -\delta \pm \sqrt{\delta^2 - \omega^2}$$

Four cases:

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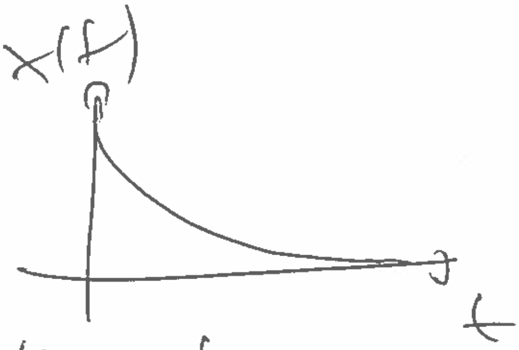
① Purely damped motion: $\delta > \omega$

$$x(t) = Ae^{\left(-\delta+\sqrt{\delta^2-\omega^2}\right)t} + Be^{\left(-\delta-\sqrt{\delta^2-\omega^2}\right)t}$$

Both roots are real & distinct and negative

(see plot)



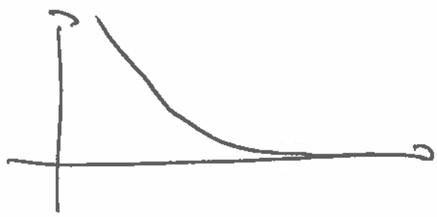
no oscillation!

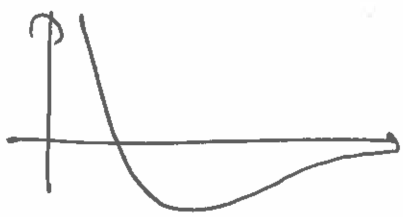
② Critically damped motion: $\delta = \omega$

repeated roots $\lambda_{1,2} = -\delta$

$$x(t) = Ae^{-\delta t} + Bte^{-\delta t}$$

(see plot)





$x \to 0$ as $t \to \infty$

At most one "overshoot"

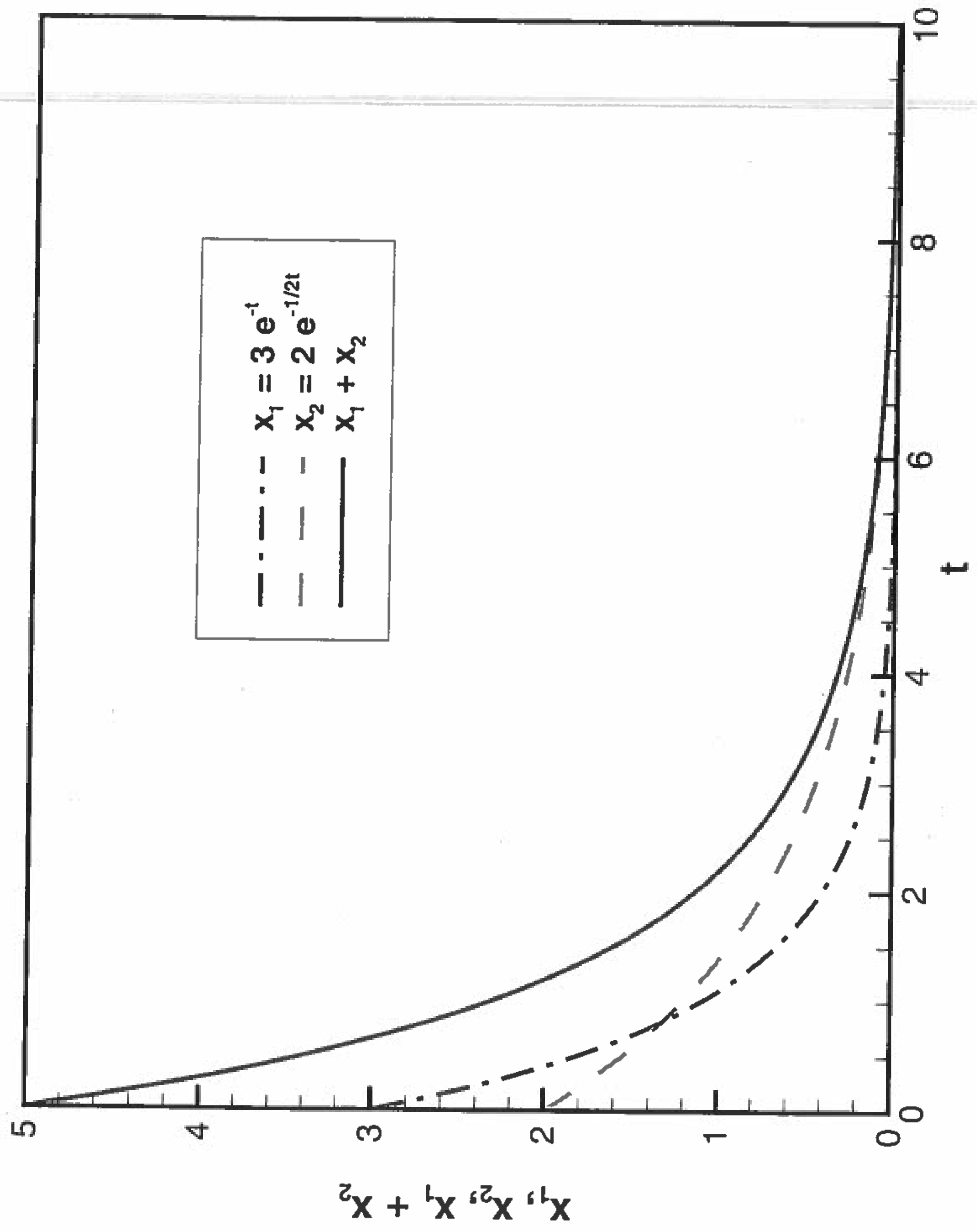


Figure 1: Illustration of a purely damped motion. The mass approaches its equilibrium position $x = 0$ monotonically.

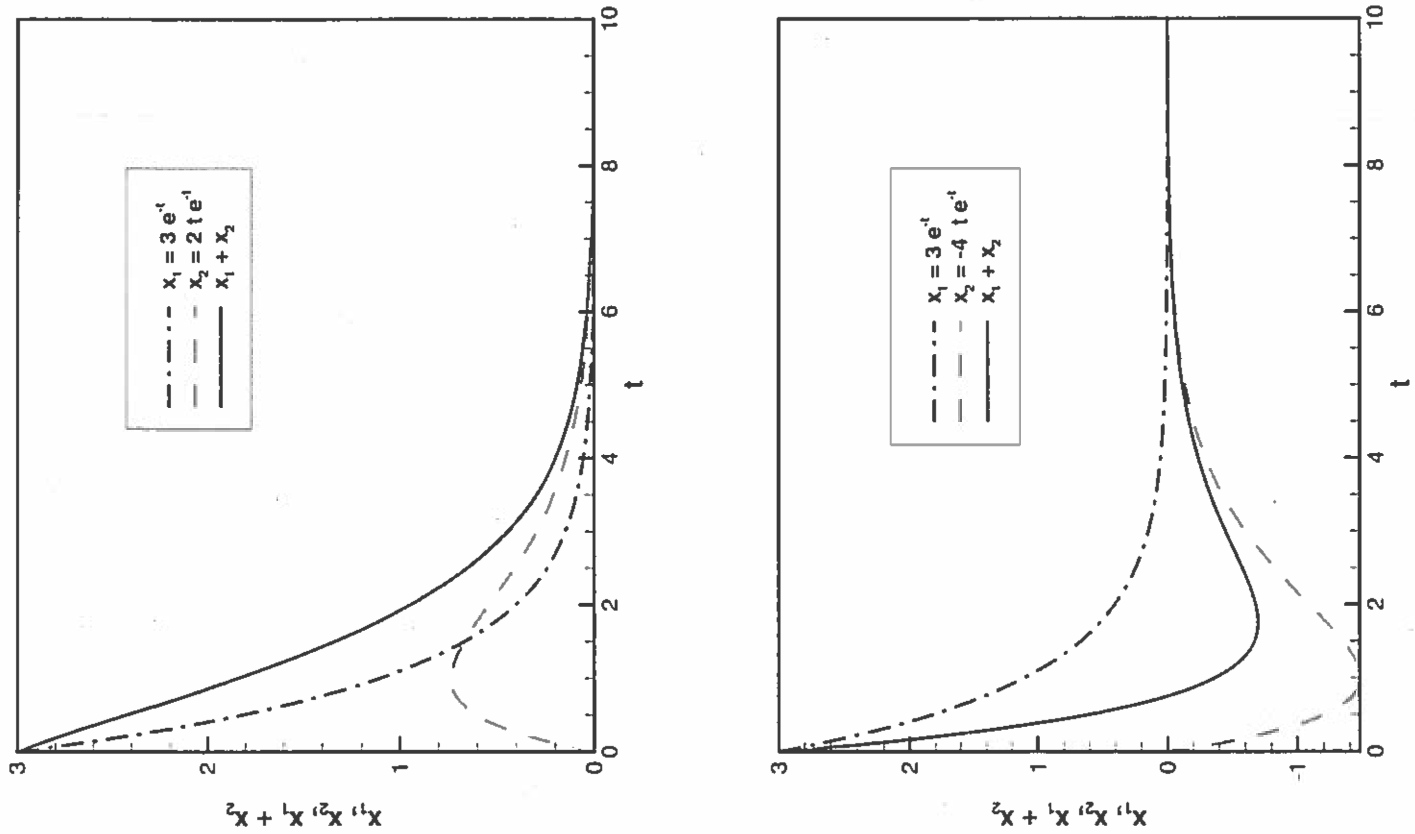


Figure 2: Illustration of critically damped motions. The mass approaches its equilibrium position, $x = 0$, with at most one "overshoot".

across $x=0$ at finite time.

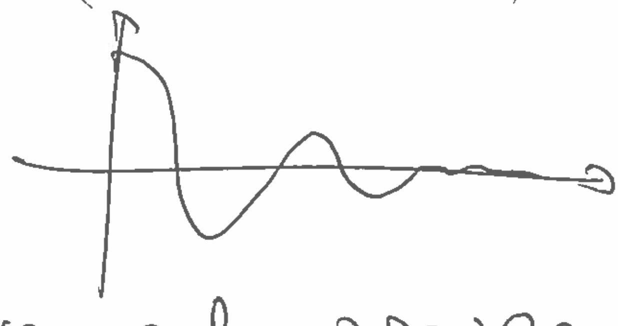
Depends on I.C.

(3) Damped oscillations: $\delta < \omega$

$$\lambda_{1,2} = -\delta \pm i\sqrt{\omega^2 - \delta^2}$$

$$x(t) = e^{-\delta t}\left(A\cos\left(\sqrt{\omega^2-\delta^2}\,t\right) + B\sin\left(\sqrt{\omega^2-\delta^2}\,t\right)\right)$$

(see plot)



damped oscillation with frequency $\sqrt{\omega^2-\delta^2}$ whose amplitude decays $\sim e^{-\delta t}$.

Note: $\frac{1}{\delta}$ = timescale it takes for the oscillation to decay.

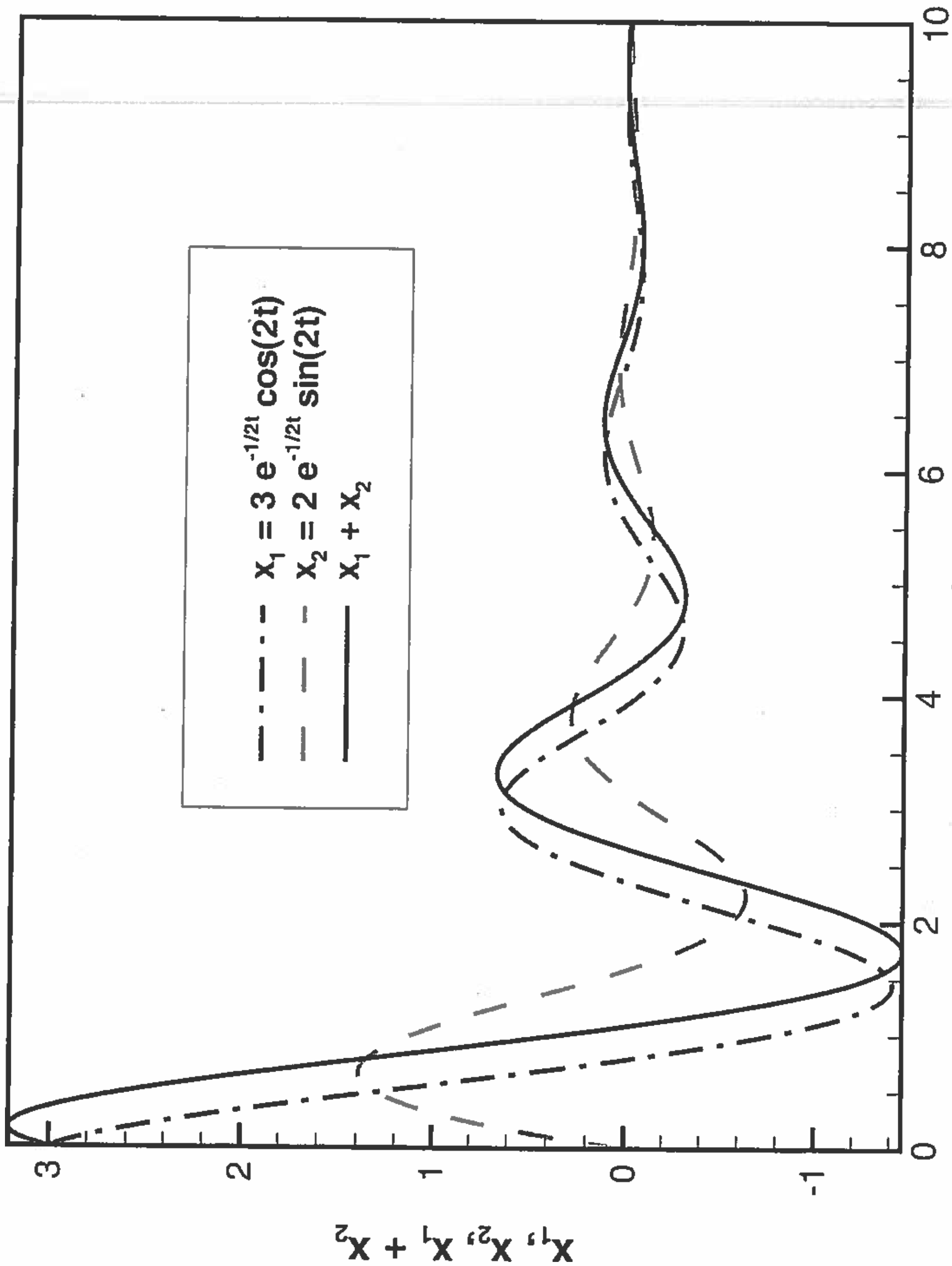


Figure 3: Illustration of a damped oscillation. The mass oscillates about its equilibrium position $x = 0$ and the amplitude of the oscillations decays exponentially.

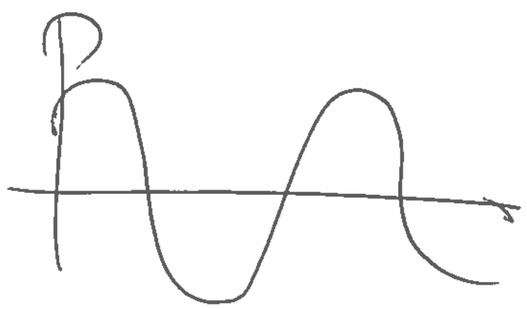
(4) Undamped oscillation: $\delta = 0$

$$\lambda_{12} = \pm i\omega$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

purely harmonic, undamped oscillation. Note: $\omega^* = \sqrt{\frac{c}{m}}$ is the eigenfrequency of the system in the absence of damping.

(see plot)



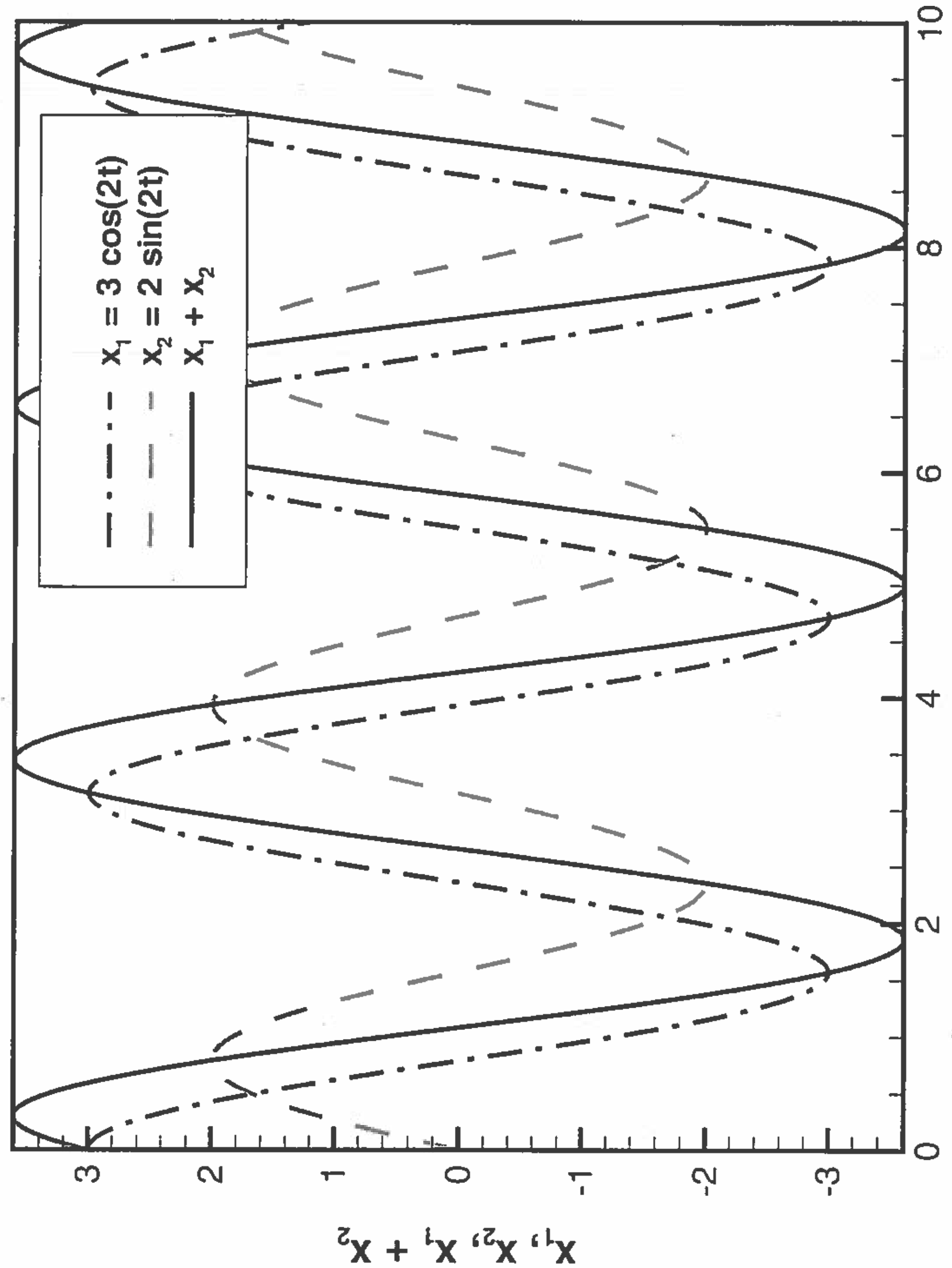


Figure 4: Illustration of an undamped oscillation. The mass performs harmonic oscillations about its equilibrium position $x = 0$.

Forced oscillations:

Periodic forcing & "resonance"

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = f(t)$$

Gen soln: Gen. soln. of the homog. eqn (= free osc.) + one particular soln. of the inhomog. ODE.

Periodic/harmonic forcing:

$$f(t) = \hat{f}\sin(\Omega t) \quad \text{or} \quad \hat{f}\cos(\Omega t)$$

Can do both at the same time by writing

$$f(t) = \hat{f}e^{i\Omega t}$$

& then extracting the real or imag. part of the soln (for cos. & sin, respectively)

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = \hat{f}\, e^{i\Omega t}$$

<u>Ansatz:</u>

$$x_p = \bar{X}\, e^{i\Omega t}$$

$$\dot{x}_p = i\Omega \bar{X}\, e^{+i\Omega t}$$

$$\ddot{x}_p = -\Omega^2 \bar{X} e^{i\Omega t}$$

into ODE:

$$\bar{X} \cancel{e^{i\Omega t}} \left(-\Omega^2 + 2\delta i\Omega + \omega^2\right) \overset{!}{=} \hat{f} \cancel{e^{i\Omega t}}$$

$$\bar{X} = \frac{\hat{f}}{(\omega^2 - \Omega^2) + i(2\delta\Omega)} \quad \text{complex}$$

$$\bar{X} = X_{real} + i X_{imag} = |\bar{X}|\, e^{i\varphi}$$

because $x_p(t) = \bar{X} e^{i\Omega t}$

the amplitude of the oscillation is controlled by $|\bar{X}|$.

$$|\bar{X}| = \frac{\hat{f}}{\sqrt{(\omega^2-\Omega^2)^2 + (2\delta\Omega)^2}}$$

$$\frac{|\bar{X}|}{\hat{f}/\omega^2} = \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2 + \left(2\left(\frac{\delta}{\omega}\right)\left(\frac{\Omega}{\omega}\right)\right)^2}}$$

ratio of the magnitude of the oscillation to the applied forcing.

ratio of forcing frequency to eigen-frequency.

ratio of damping to spring stiffness

Normalised amplitude of the oscillation of the harmonically forced mechanical oscillator