$$\ddot{y} + 4y = \cos(3t) + 2\sin(t)$$

$$= A_1 r_1(t) + A_2 r_2(t)$$

Eg. $A_1 = 1$, $r_1(t) = \cos(3t)$

$A_2 = 2$, $r_2(t) = \sin(t)$

$$y = y_p + \underbrace{y_H}_{}$$

$\to$ gen. soln. of homog. ODE

$$y_H = \hat{A}\, y_1(t) + \hat{B}\, y_2(t)$$

Here: (EXERCISE)

$$y_H = \hat{A}\sin(2t) + \hat{B}\cos(2t)$$

<u>y_p:</u> <u>Ansatz:</u>

$$y_p = \underbrace{A \underbrace{\cos(3t)}_{r_1(t)} + B}_{A,\,B \text{ undet. coeffns}} \underbrace{\sin(t)}_{r_2(t)}$$

into ODE

$$\underbrace{\cos(3t)}_{r_1(t)}\left(-9A+4A-1\right)+\underbrace{\sin(t)}_{r_2(t)}\left(-B+4B-2\right)=0$$

$\forall t$!

& $r_1(t)$ & $r_2(t)$ lin. indep.

$\Rightarrow$ $A=-1/5$ $\quad B=2/3$

Example:

$$\ddot{y}+3\dot{y}+y=4+2t^2$$

$A_1=4 \qquad r_1(t)=1$

$A_2=2 \qquad r_2(t)=t^2$

Ansatz:

$$y_p=c_1\cdot\underbrace{1}_{r_1(t)}+c_2\underbrace{t^2}_{r_2(t)}+c_3t$$

$$\dot{y}_p=2c_2t+c_3$$

$$\ddot{y}_p=2c_2$$

into ODE:

$$\underbrace{2c_2}_{\ddot{y}} + 3\underbrace{(2c_2 t + c_3)}_{\dot{y}} + \underbrace{c_1 + c_2 t^2 + c_3 t}_{y} \overset{!}{=} 4 + 2t^2$$

collect lin. indep. fcts. Here: powers of t

$$(2c_2 + c_1 - 4 + 3c_3) +$$
$$(6c_2 + c_3)\, t +$$
$$(c_2 - 2)\, t^2 \overset{!}{=} 0 \quad \forall t$$

$$\left.\begin{array}{l} c_2 = 2 \\ c_2 = 0 \end{array}\right\}$$

☹

Reason: differentiation of t^2 gives rise to t
$\Rightarrow$ include in ansatz!

Now solve the eqns: (4)

$$\underline{\underline{c_2 = 2}}$$

$$6c_2 + c_3 = 0 \Rightarrow \underline{\underline{c_3 = -12}}$$

$$2c_2 + c_1 - 4 + 3c_3 = 0$$

$$\Rightarrow \underline{\underline{c_1 = 36}}$$

$$\underline{\underline{y_p = 36 - 12t + 2t^2}}$$

<u>Example:</u>

$$\ddot{y} + 3\dot{y} = \underbrace{1}_{r_1(t)} + \underbrace{9t^2}_{r_2(t)}$$

<u>Ansatz:</u>

$$\hat{y}_p = c_1 + c_2 t + c_3 t^2$$

$$\dot{\hat{y}}_p = c_2 + 2c_3 t$$

$$\ddot{\hat{y}}_p = 2c_3$$

into ODE:

$$\underbrace{2c_3}_{\ddot{y}} + 3\underbrace{(c_2 + 2c_3 t)}_{\dot{y}} \stackrel{!}{=} 1 + 9t^2$$

collect coeffns:

$$(2c_3 + 3c_2 - 1) +$$

$$+ (6c_3)t +$$

$$+ \underbrace{(-9)}_{\text{has to be zero}} t^2 = 0 \quad \forall t$$

has to be zero ☹

Recall: Method does not work in its simplest form if a term on RHS is a soln. of the homog. ODE.

Fix: Multiply ansatz by t^m where m is

6

Smallest integer for which

$t^m \hat{y}_p$ does not have

terms that are solns of

homog. ODE.

Here: try $m=1$

$$y_p = c_1 t + c_2 t^2 + c_3 t^3$$

$$\dot{y}_p = c_1 + 2c_2 t + 3c_3 t^2$$

$$\ddot{y}_p = 2c_2 + 6c_3 t$$

into ODE:

$$\underbrace{2c_2 + 6c_3 t}_{\ddot{y}} + 3\underbrace{(c_1 + 2c_2 t + 3c_3 t^2)}_{\dot{y}}$$

$$\stackrel{!}{=} 1 + 9t^2$$

collect coeffns.:

$$\underbrace{(2c_2 + 3c_1 - 1)}_{=0} + \overbrace{(6c_3 + 6c_2)}^{=0} t +$$

$$+ \underbrace{(9c_3 - 9)}_{=0} t^2 \stackrel{!}{=} 0 \quad \forall t$$

$$C_3 = 1$$
$$C_2 = -1$$
$$C_1 = 1$$

$$\Rightarrow \underline{\underline{y_p = t - t^2 + t^3}}$$

<u>Finally</u>: The method only works for certain RHSs.

<u>Counterexample</u>:

$$r(t) = \log(t) \Rightarrow t^{-1}$$
$$\Downarrow$$
$$t^{-2}$$
$$\Downarrow$$
$$\ldots \; t^{-4} \Leftarrow t^{-3}$$

Most general form of the RHS for which the method works is fcts of the form

$$r(x) = \underbrace{P(x)}_{\text{polynomials}} e^{mx} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (nx)$$

$\Downarrow$

c.f. Taylor expansions

$\Longrightarrow$ Series solutions etc.

$\Downarrow$

2nd year!

Nonlinear 2nd order ODEs

2 special cases:

1) ODE does not depend on y

$$\boxed{y'' = f(x, \cancel{y}, y') = f(x, y')}$$

This is actually a 1st order ODE for y'.

Substitution:

$$u(x) = y'(x)$$

$$\boxed{\frac{du}{dx} = f(x, u)}$$

Solve for $u(x)$; introduces one constant

Then solve:

$$\boxed{\frac{dy}{dx} = u(x)}$$

Another 1st order ODE for $y(x)$

$\Rightarrow$ second constant.

Example:

$$3(y')^2 y'' = 1$$

Subst: $y' = u \quad y'' = u'$

$$3u^2 \frac{du}{dx} = 1 \qquad \text{Sep. of vars}$$

$$\int 3u^2 \, du = \int dx$$

$$u^3 = x + C$$

$$u = \boxed{(x+C)^{1/3} = \frac{dy}{dx}}$$

Sep again or just integrate

$$y(x) = \frac{3}{4}(x+C)^{4/3} + D.$$

② Autonomous ODEs

$$y'' = f(\cancel{x}, y, y') = f(y, y')$$

indep. variable, x, does not appear explicitly.

Can again be reduced to a 1^{st} order ODE for ~~$v(x) = y'(x)$~~ $y' = v$ but we regard v as a fct of y:

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\underbrace{\frac{dy}{dx}}_{v} = v\frac{dv}{dy}$$

into ODE:

$$v\frac{dv}{dy} = f(y, v)$$ — 1^{st} order ODE for $v(y)$!

Solve for $v(y)$ then solve

$$\frac{dy}{dx} = v(y(x))$$ — 1^{st} order ODE for $y(x)$.

Example:

$$y y'' - 2(y')^2 + 2y' = 0$$

Note: $y = 0$ is a soln!

Set $u = y'$

$$y'' = u \frac{du}{dy}$$

into ODE:

$$y u \frac{du}{dy} - 2u^2 + 2u = 0$$

$$u \underbrace{\left(y \frac{du}{dy} - 2u + 2 \right)}_{\Rightarrow 0} = 0$$

$\hookrightarrow$ $u = \frac{dy}{dx} = 0$ is a soln.

$\Rightarrow$ $y = \text{const}$ is a soln.

2nd family of solns:

$$y \frac{du}{dy} = 2u - 2$$

Sep. vars.

$$\int \frac{1}{u-1}\,du = \int \frac{2}{y}\,dy$$

$$\ln|u-1| = 2\ln|y| + \hat{C}$$

$$= \ln y^2 + \ln D$$

$$= \ln(Dy^2)$$

$$u(y) = 1 + Dy^2 = \frac{dy}{dx}$$ EXERCISE mod?

Sep. again

$$\int dx = \underbrace{\int \frac{dy}{1+Dy^2}}$$

$x =$ Standard integral

$$\int \frac{dy}{a^2+y^2} = \frac{1}{a}\arctan\left(\frac{y}{a}\right)$$

$\vdots$

$$y = \frac{1}{\sqrt{D}}\tan\left(\sqrt{D}\,x + C\right)$$ EXERCISE