$$y'' + p y' + q y = r(x)$$

const

$$= A_1 r_1(x) + \dots + A_n r_n(x)$$

$r_i(x)$ lin. indep.

Ansatz:

$$y_p = C_1 r_1(x) + \dots + C_n r_n(x)$$

undetermined coeffns

~~Feed~~ "into ODE"

Modific 1: new lin. indep. fcts $\Rightarrow$ add them

Modific 2: Any fct solves homog. ODE $\Rightarrow$ multiply by x^m

$$\ddot{y} + 4y = \underbrace{\cos(3t)}_{r_1(t)} + 2\underbrace{\sin(t)}_{r_2(t)}$$

$$y_H = \hat{A}\sin(2t) + \hat{B}\cos(2t)$$

Ansatz:

$$y_P = A\cos(3t) + B\sin(t)$$

in DGL

$$\cos(3t)\underbrace{\left(5A+1\right)}_{0} + \sin t\underbrace{\left(3B-2\right)}_{0} = 0 \quad \forall t$$

$$A = -1/5 \qquad B = 2/3$$

Example:

$$\ddot{y} + 3\dot{y} + y = 4 + 2t^2$$

$$A_1 = 4$$
$$r_1(t) = 1$$
$$A_2 = 2$$
$$r_2(t) = t^2$$

Ansatz:

$$y_p = C_1 \cdot \underbrace{1}_{r_1(t)} + C_2 \underbrace{t^2}_{r_2(t)} + C_3 t$$

$$\dot{y}_p = +2C_2 t + C_3$$

$$\ddot{y}_p = 2C_2$$

into ODE

$$\underbrace{2C_2}_{\ddot{y}} + 3\underbrace{(2C_2 t + C_3)}_{\dot{y}} \underbrace{+ C_1 + C_2 t^2 + C_3 t}_{y}$$

$$\overset{!}{=} 4 + 2t^2$$

Coeff of lin. indep. fcts (powers of t)

$$\overbrace{(2c_2+3c_3+c_1-4)}^{=0}\cdot 1+\overbrace{(6c_2+c_3)}^{=0}\,t+\underbrace{(c_2-2)}_{=0}\,t^2 \overset{!}{=} 0 \qquad \forall t$$

$\underline{t^2}$: $c_2 = 2$ ✓

$\underline{t}$: $\cancel{c_2 = 0}$ $\lightning$

$6c_2 + c_3 = 0 \Rightarrow c_3 = -12$

$\underline{t^0}$: $c_1 = 4 - 3c_3 - 2c_2 = 36$

$$y_p = 36 - 12t + 2t^2$$

EXAMPLE:

$$\ddot{y} + 3\dot{y} = \underbrace{1}_{r_1(t)} + \underbrace{9t^2}_{r_2(t)}$$

$$\hat{y}_p = C_1 + C_2 t + C_3 t^2$$

$$\dot{\hat{y}}_p = C_2 + 2C_3 t$$

$$\ddot{\hat{y}}_p = 2C_3$$

into ODE:

$$\underbrace{2C_3}_{\ddot{y}} + 3\underbrace{\left(C_2 + 2C_3 t\right)}_{\dot{y}} \overset{!}{=} 1 + 9t^2$$

collect lin. indep. fcts:

$$\underbrace{\left(2C_3 + 3C_2 - 1\right)}_{0} + \underbrace{\left(6C_3\right)}_{0} t + \underbrace{\left(-9\right)}_{0} t^2 \overset{!}{=} 0$$

Doesn't work because the constant f.c.t. $r_1(t) = 1$ is a soln. of the homog. ODE.

Soln: multiply ansatz by t,

$$y_p = t\,\hat{y}_p$$

$$y_p = c_1 t + c_2 t^2 + c_3 t^3$$

$$\dot{y}_p = c_1 + 2c_2 t + 3c_3 t^2$$

$$\ddot{y}_p = 2c_2 + 6c_3 t$$

into ODE:

$$\underbrace{(2c_2 + 6c_3 t)}_{\ddot{y}} + 3\underbrace{(c_1 + 2c_2 t + 3c_3 t^2)}_{\dot{y}} \overset{!}{=} 1 + 9t^2$$

collect:

$$\underbrace{(2c_2+3c_1-1)}_{0} \cdot 1 + \underbrace{(6c_3+6c_2)}_{0}\, t + \underbrace{(9c_3-9)}_{0}\, t^2 \overset{!}{=} 0$$

$c_3 = 1$

$c_2 = -1$

$c_1 = 1$

$$y_p = t - t^2 + t^3$$

SADLY this method only works for a small number of fcts $r_i(x)$.

For instance it doesn't work for
$$r(x) = \log(x)$$

(8

$$y_p = C_1 \log(x) + C_2 x^{-1} + C_2 x^{-2} + C_3 x^{-3}$$

$$y_p' = C_1 \frac{1}{x} + (-C_2 x^{-2})$$

$$y_p'' = -2C_1 \frac{1}{x^2} + \dots x^{-3}$$

This never stops – we keep adding more & more lin. indep. fcts

In fact, it only works for

$$r(x) = \underbrace{p(x)}_{\text{polynomial of } x} e^{mx} \begin{Bmatrix} \cos(nx) \\ \sin(nx) \end{Bmatrix}$$

<u>Alternatives:</u>

- Method of variations of parameters
- Power series expansion.

Nonlinear ODEs

2 special cases:

① ODE does not depend on y

$$y'' = f(x, \cancel{y}, y')$$

This is a 1^{st} order ODE for y': So do substitution

$$y' = v \quad \text{into ODE}$$

$$v' = f(v, x)$$

- solve this for $v(x)$; one const. of integration
- solve:

$$y' = \frac{dy}{dx} = v(x)$$

for $y(x)$; second const. of integration

Example:

$$3(y')^2 y'' = 1$$

Subst: $y' = v$

$$3v^2 v' = 1$$

$$3v^2 \frac{dv}{dx} = 1$$

$$\int 3v^2 dv = \int dx$$

$$v^3 = x + C$$

$$v = \boxed{(x+C)^{1/3} = \frac{dy}{dx}}$$

$$\int dy = \int (x+C)^{1/3} dx$$

$$y = \frac{3}{4}(x+C)^{4/3} + D$$

② Autonomous ODES

$$y'' = f(\cancel{x}, y, y')$$

indep variable, x, does not appear explicitly.

Can again be reduced to 1^{st} order ODE for $y' = v$ if we regard v as a fct of y rather than x.

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\underbrace{\frac{dy}{dx}}_{v}$$

$$v\frac{dv}{dy} = f(y, v)$$ is a 1^{st} order ODE for $v(y)$.

Then solve

$$\frac{dy}{dx} = v(y(x))$$ for $y(x)$.

Example:

$$y y'' - 2(y')^2 + 2y' = 0$$

Note: $y = 0$

$$y' = v$$
$$y'' = v \frac{dv}{dy}$$

$$y v \frac{dv}{dy} - 2v^2 + 2v = 0$$

$$v \underbrace{\left(y \frac{dv}{dy} - 2v + 2 \right)} = 0$$

$v = 0 \Rightarrow y = \text{const.}$ is a soln.

$$y \frac{dv}{dy} = 2v - 2 = 2(v-1)$$

$$\int \frac{dv}{v-1} = \int \frac{2}{y} dy$$

$$\ln|v-1| = 2\ln|y| + \hat{C}$$

$$= \ln y^2 + \ln D$$

$$\ln|v-1| = \ln(Dy^2)$$

$$v(y) = \boxed{1 + Dy^2 = \frac{dy}{dx}}$$

$$\int dx = \int \frac{1}{1+Dy^2}\,dy$$

$$x + \hat{\hat{C}} = \frac{1}{\sqrt{D}} \arctan(\sqrt{D}\,y)$$

$$y = \frac{1}{\sqrt{D}} \tan(\sqrt{D}\,x + C)$$

$$D\hat{\hat{C}}\sqrt{D}$$