$$y'' + p y' + q y = r(x)$$ 1

const

$$y = y_p + A y_1 + B y_2$$

$$y_{1,2} \sim e^{\lambda x}$$

char. poly: $\lambda^2 + p\lambda + q = 0$

---

y_p ?

given!

$$y'' + p y' + q y = A e^{ax}$$

Ansatz: $y_p = C e^{ax}$

into ODE

$$C \cancel{e^{ax}} (a^2 + pa + q) = A \cancel{e^{ax}}$$

$$C = \frac{A}{a^2 + pa + q}$$

but what happens if

$a^2 + pa + q = 0$?

This happens if: a is a root (2) of the char. poly. and/or if e^{ax} is a soln. of the homog. ODE.

In this case:

Try: $y = C' x e^{ax}$

$y' = C' e^{ax}(1+ax)$

$y'' = C' e^{ax}(a(2+ax))$

into ODE:

$$\cancel{C' e^{ax}} \left\{ \underbrace{a(2+ax)}_{y''} + p\underbrace{(1+ax)}_{y'} + qx \right\} \overset{!}{=} A\cancel{e^{ax}}$$

$$C'\left(2a + \underline{a^2 x} + p + \underline{pax} + \underline{qx}\right) \overset{!}{=} A$$

$$C'\left(x\underbrace{\left(a^2+pa+q\right)}_{0} + 2a + p\right) \overset{!}{=} A$$

$$C = \frac{A}{2a+p}$$

This works unless $2a+p=0$ which happens if a is a repeated root of the char. poly. and/or xe^{ax} is also a soln. of the homog. ODE.

In that case try:

$$y = Cx^2 e^{ax}$$

$$y' = Ce^{ax}(2x+ax^2)$$

$$y'' = Ce^{ax}(2+4ax+a^2x^2)$$

into ODE:

$$C\cancel{e^{ax}}\left\{\underbrace{2+\cancel{4ax}+\underline{a^2x^2}}_{y''}+p\underbrace{(\cancel{2x}+\underline{ax^2})}_{y'}+\underbrace{\underline{qx^2}}_{y}\right\} \overset{!}{=} A\cancel{e^{ax}}$$

collect powers of x:

$$C\left(x^2\underbrace{(a^2+pa+q)}_{0}+x\underbrace{(4a+2p)}_{0}+2\right)\stackrel{!}{=}A$$

$$\underline{\underline{C=\frac{1}{2}A}}$$

& this definitely works.

General procedure for

$$y''+py'+qy=Ae^{ax}$$

① Solve the char. poly.

$$\lambda^2+\lambda p+q=0 \Rightarrow \lambda_{12}$$

fundam. solns y_1, y_2

② • If $a\neq\lambda_1$ & $a\neq\lambda_2$:

$$y_p=Ce^{ax}$$

• If $a=\lambda_1$ or $a=\lambda_2$ but $\lambda_1\neq\lambda_2$

$$y_p=Cxe^{ax}$$

· If $a = \lambda_1 = \lambda_2$

$$y_p = C x^2 e^{ax}$$

Have classified special cases in terms of roots of the char. poly.

<u>Alternatively:</u>
Special cases arise if $r(x)$ is a soln. of the homog. ODE.

General approach:

$$y'' + py' + qy = A_1 r_1(x) + \dots + A_n r_n(x)$$

where the $r_i(x)$ are lin. indep. w.l.o.g.

<u>Fact:</u> If $y_1(x)$ is a solution of

$$y'' + py' + qy = r_1(x)$$

and $y_2(x)$ is a soln. of

$$y'' + py' + qy = r_2(x)$$

then $A_1 y_1(x) + A_2 y_2(x)$
is a solution of of

$$y'' + py' + qy = A_1 r_1(x) + A_2 r_2(x)$$

(EXERCISE)

Ansatz:

$$y_p = C_1 r_1(x) + \dots + C_n r_n(x)$$

Plan: insert into ODE, collect the linearly indep. fcts & set their coefficients to zero $\Rightarrow$ n eqns for n unknowns $C_1, \dots, C_n$

Modification: If the differentiation of any of the $r_i(x)$ creates new linearly indep. fcts then include them too:

$$\hat{y}_p = C_1 r_1(x) + \dots + C_n r_n(x) + D_1 r_1'(x) + \dots + D_n r_n'(x) + E_1 r_1''(x) + \dots + E_n r_n''(x)$$

only include new terms.

what can go wrong?

If any of the fcts in $\hat{y}_p$ are solns. of the homog. ODE this won't work:

$$y'' + py' + q = \underbrace{A\, r(x)}_{\text{given}}$$

Ansatz: $y_p = C\, r(x)$

into ODE:

$$C\underbrace{\left(r'' + pr' + qr\right)}_{=0} \overset{!}{=} A\, r(x)$$

In that case try:

$$y_p = C' x\, r(x)$$
$$y_p' = C'(r + x r')$$
$$y_p'' = C'(2r' + x r'')$$

into ODE:

$$C'\Big(\underbrace{2r' + x r''}_{y_p''} + p\underbrace{(r + x r')}_{y_p'} + q\underbrace{x r}_{y_p}\Big) \stackrel{!}{=} Ar$$

$$C'\Big(x\underbrace{(r'' + p r' + q r)}_{=0} + 2r' + p r\Big) \stackrel{!}{=} Ar$$

This works unless $2r' + pr = 0$

This happens if $y_p = C' x\, r(x)$ is also a soln. of the homog. ODE.

In that case:

$$y_p = C' x^2 r(x)$$
$$y_p' = C'(2xr + x^2 r')$$
$$y_p'' = C'(2r + 4xr' + x^2 r'')$$

inb ODE:

$$C'\Big\{\underbrace{2r + 4xr' + x^2 r''}_{y_p''} + p\,\underbrace{(2xr + x^2 r')}_{y_p'} + \underbrace{q x^2 r}_{y_p}\Big\} \overset{!}{=} A\,r(x)$$

$$C'\Big\{x^2\overbrace{(r'' + pr' + qr)}^{0} + x\overbrace{(4r' + 2pr)}^{0} + 2r\Big\} \overset{!}{=} A\,r$$

$$\Rightarrow \quad \underline{\underline{C' = \frac{1}{2} A}}$$

Modification 2: If one of the terms in $\hat{y}_p$ is a solution of the homog. ODE, multiply by x^m where m is the smallest integer for which none of the terms are solutions of the homog. ODE.

If this introduces any new lin. indep. fcts, introduce them too.

Example:

$$\ddot{y} + 4y = \underbrace{\cos(3t)}_{r_1(t)} + 2\,\underbrace{\sin(t)}_{r_2(t)}$$

① y_H: $\ddot{y} + 4y = 0$

$$\lambda^2 + 4 = 0$$

$$\lambda_{12} = \pm 2i$$

$$y_H = \hat{A}\,\underbrace{\sin(2t)}_{y_1} + \hat{B}\,\underbrace{\cos(2t)}_{y_2}$$

② y_P:

$r_1(t) = \cos(3t)$; $A_1 = 1$

$r_2(t) = \sin(t)$; $A_2 = 2$

[or $r_2(t) = \frac{1}{2}\sin(t)$; $A_2 = 4$]

Ansatz:

$$y_p = \underset{\downarrow\, C_1}{A} \underbrace{\cos(3t)}_{r_1(t)} + \underset{\downarrow\, C_2}{B} \underbrace{\sin(t)}_{r_2(t)}$$

$$\dot{y}_p = -3A\sin(3t) + B\cos(t)$$

$$\ddot{y}_p = -9A\cos(3t) - B\sin(t)$$

into ODE

$$\underbrace{-9A\cos(3t) - B\sin(t)}_{\ddot{y}} + 4\underbrace{(A\cos(3t) + B\sin(t))}_{y} \overset{!}{=} \cos(3t) + 2\sin(t)$$

collect lin. indep. fcts.

$$\cos(3t)\underbrace{(-9A + 4A - 1)}_{\overset{!}{=}0} + \sin(t)\underbrace{(-B + 4B - 2)}_{\overset{!}{=}0} = 0 \quad \forall t$$

Now recall: $\cos(3t)$ & $\sin(t)$ are lin. indep.

$A = -1/5$; $B = 2/3$

Gen. soln:

$$y = -\frac{1}{5}\cos(3t) + \frac{2}{3}\sin(t) + \hat{A}\sin(2t) + \hat{B}\cos(2t).$$