Existence and uniqueness theorem for 1st order ODEs

Consider the first-order ODE in its explicit form

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

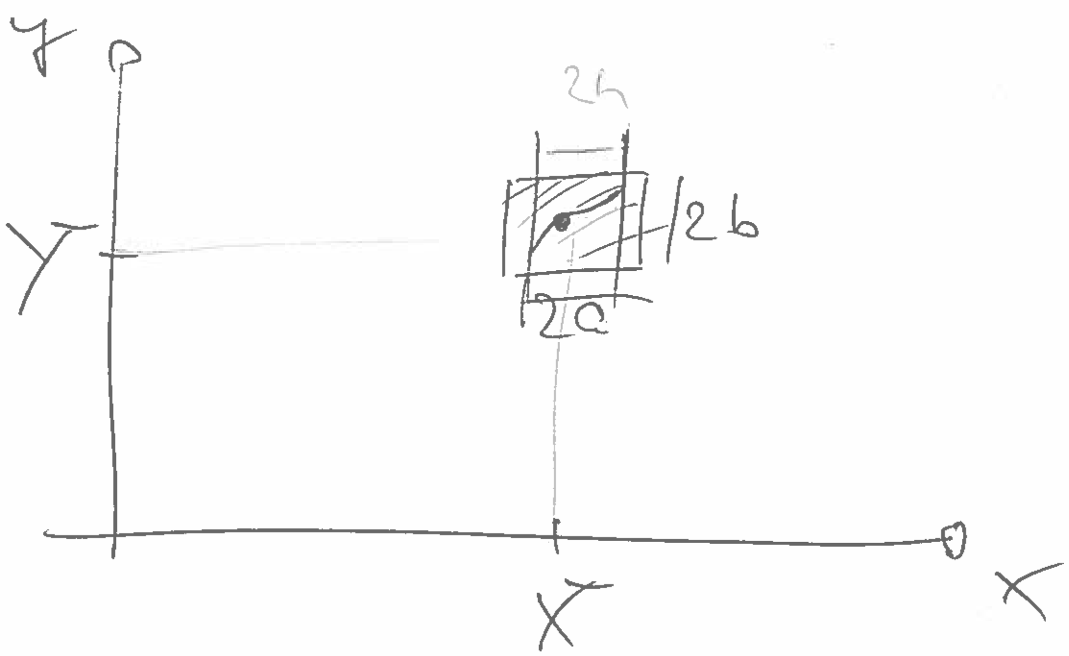
subject to the initial condition

$$y(X) = Y, \tag{2}$$

where the constants X and Y are given.

Theorem

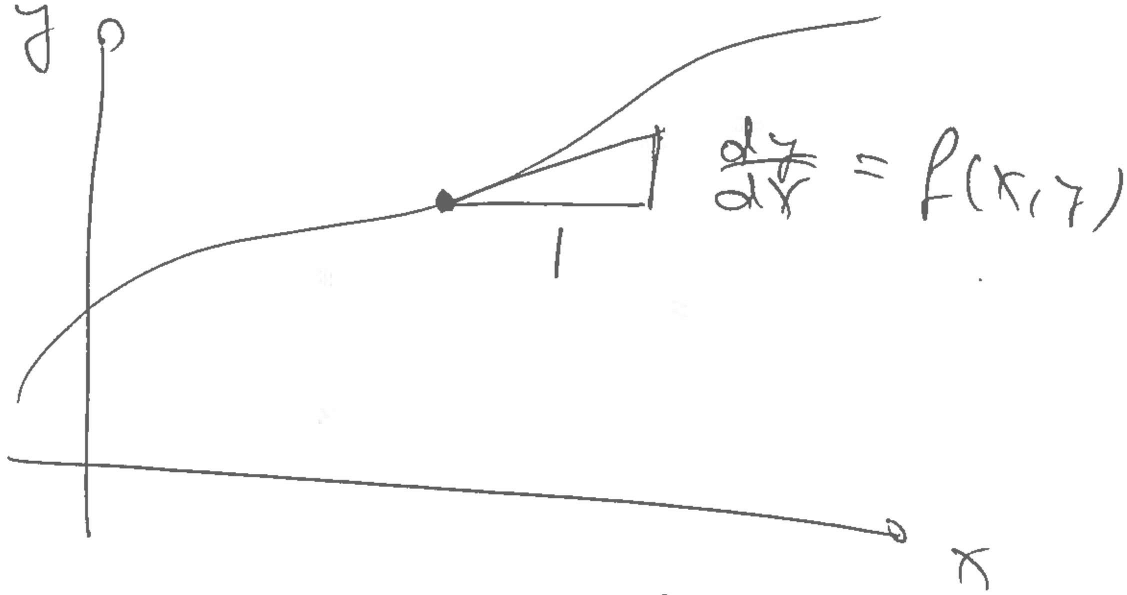
> If $f(x, y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous functions of x and y in a region $0 < |x - X| < a$ and $0 < |y - Y| < b$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.



First order ODEs

$$\boxed{y' = f(x,y)}$$

I Graphical approach



$f(x,y)$ defines the slope of the solution.

Def: The <u>direction field</u> of the ODE $y' = f(x,y)$ is the set of all vectors that have the same direction as:

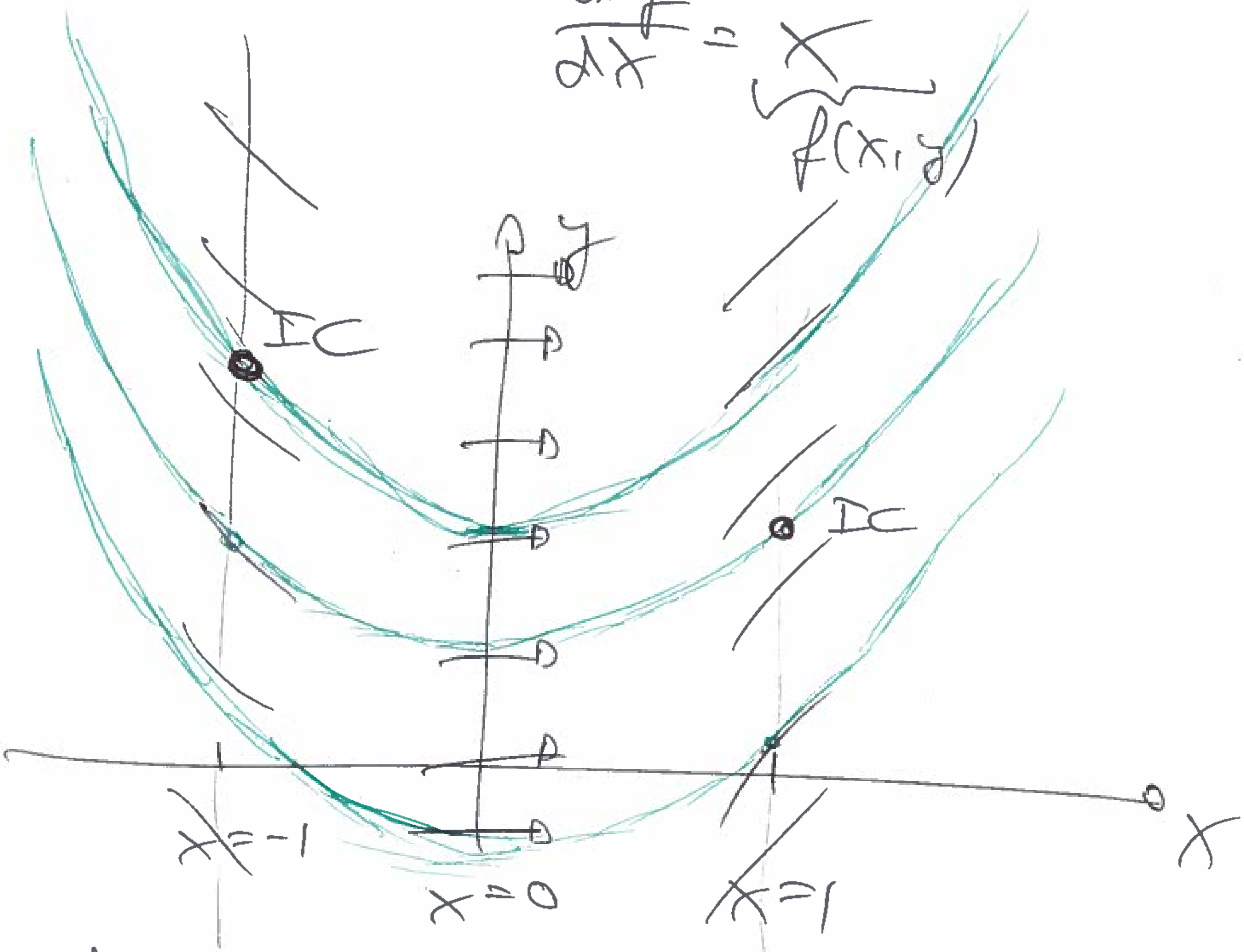
$$\underline{d} = \begin{pmatrix} 1 \\ \frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} 1 \\ f(x,y) \end{pmatrix}$$

Def: Integral curves are

curves that are everywhere tangential to the direction field. Each integral curve represents a soln. of the ODE.

Example:

$$\frac{dy}{dx} = \underbrace{x}_{f(x,y)}$$



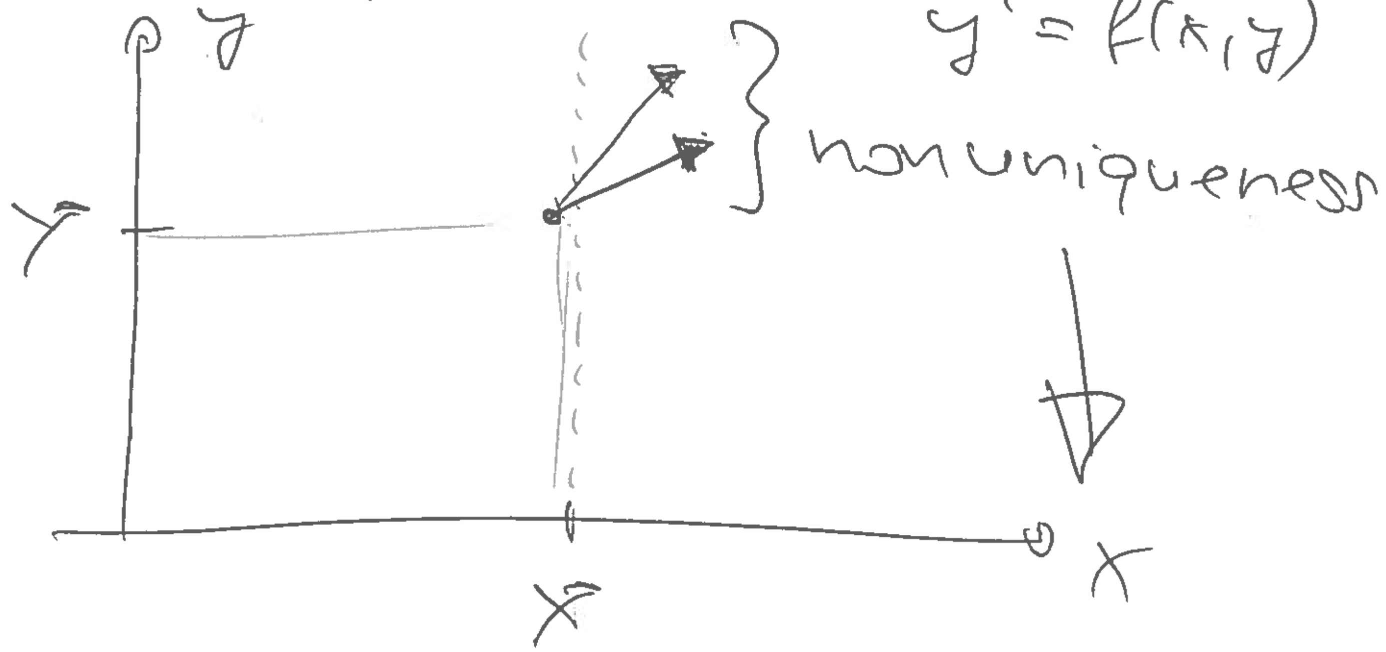
Note: • structure of soln is clear from sketch.
• soln exists everywhere & is unique (IC)!

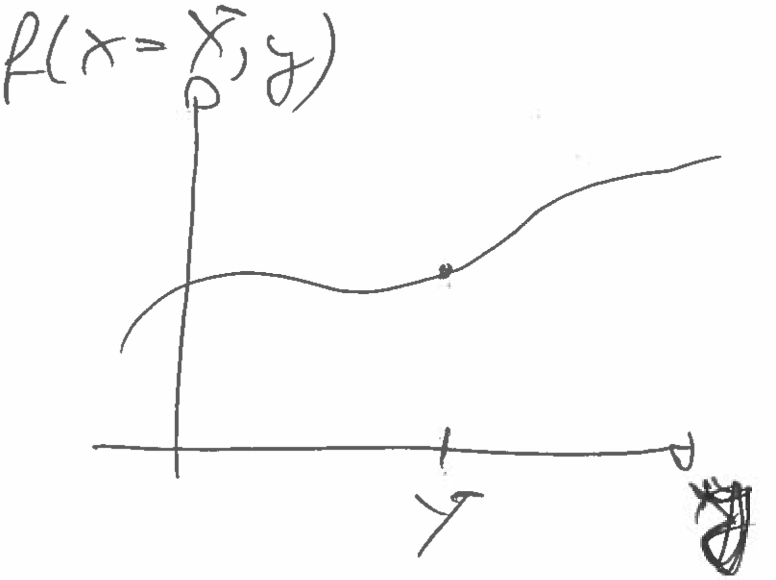
In fact:

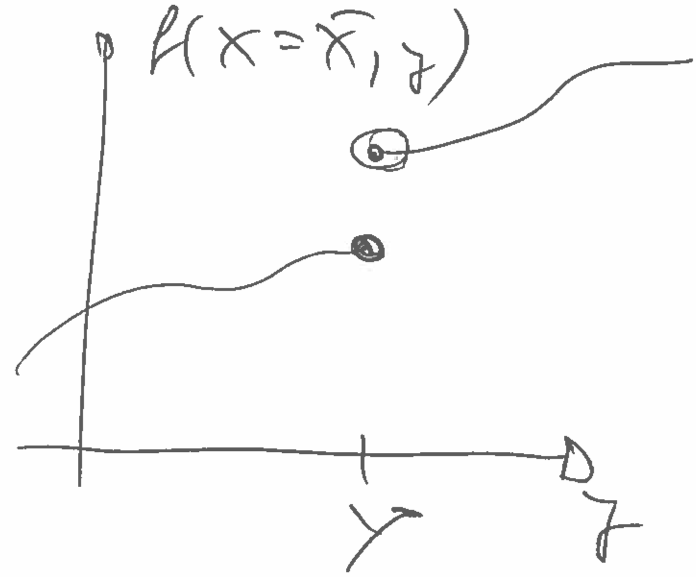
$$y = \frac{1}{2}x^2 + C$$

is the soln.

Question: How can the solution to an ODE not be unique?

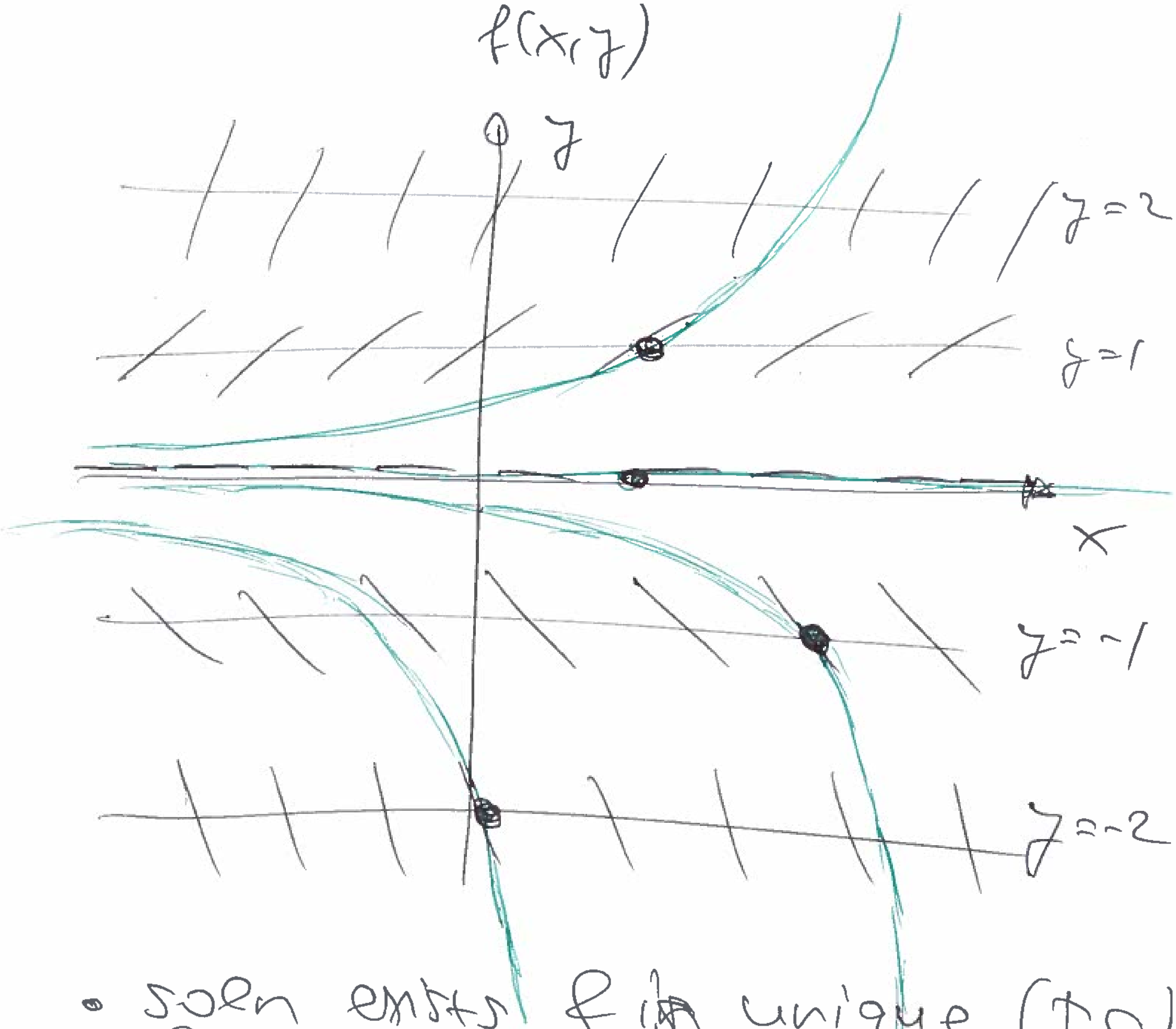






Example:

$$\frac{dy}{dx} = \underbrace{y}_{f(x,y)}$$



- soln exists & is unique (IC) for all values of x
- As $x \to -\infty$ all solns asymptote to $y=0$

Observation:

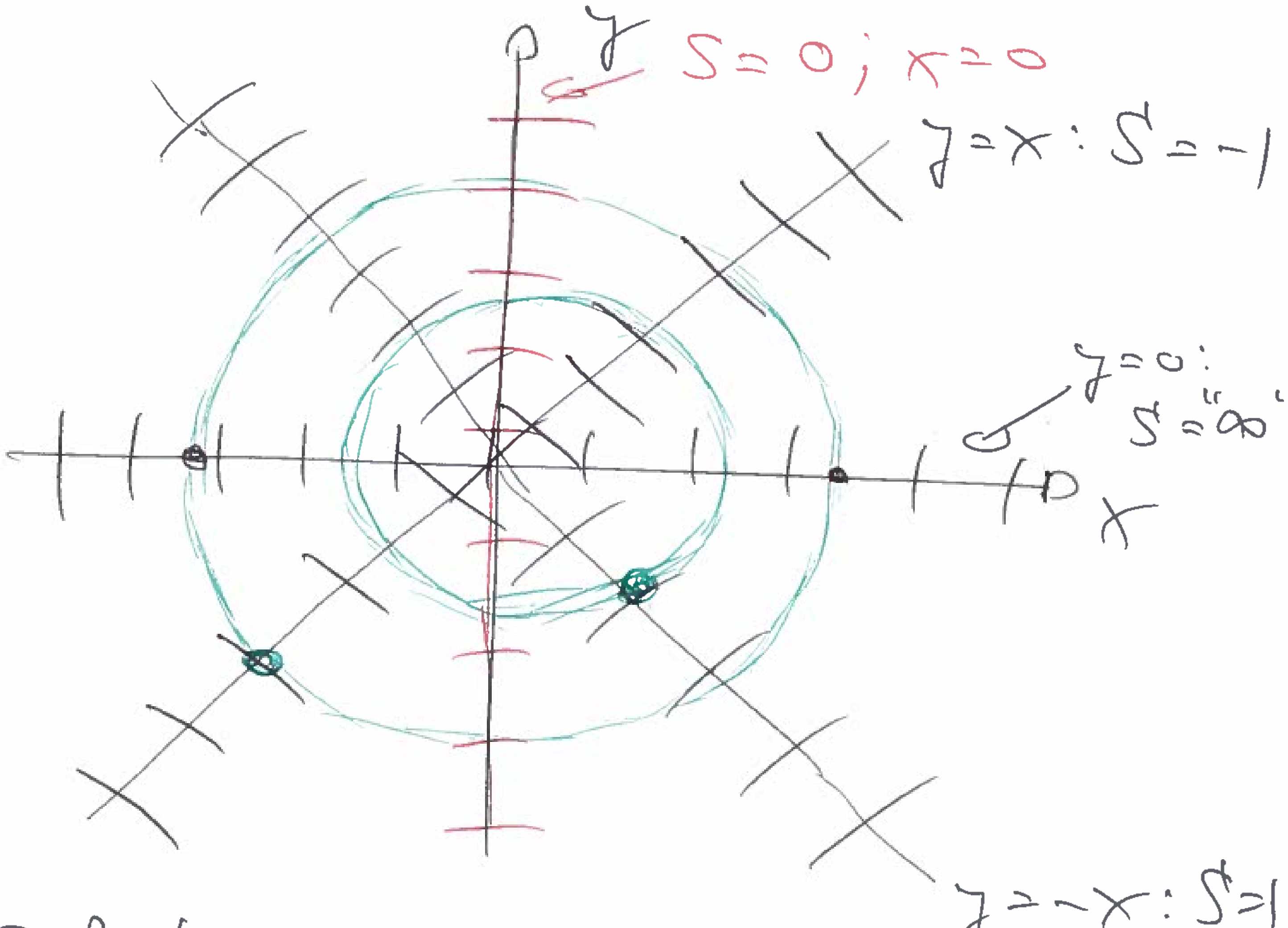
when sketching direction fields, it is helpful to identify so-called iso-clines = lines in the x-y-plane along which $\frac{dy}{dx}$ (= slope of the solution) is constant.

Example:

$$y' = -\frac{x}{y} \; ; \; y y' = -x$$

(nonlin)

slope $s = -\frac{x}{y}$ ⟶ defines the iso-cline



- solutions are circular arcs
- E & U "obvious": exactly one curve goes through each IC: $(\bar{x}, \bar{y})$ & the solution exists unless $y = 0$ (or $\bar{y} = 0$)