INTRODUCTION

Notation, Definitions and "What are the issues?"

The Derivative

Given a function

$$y(x)$$

where

- x is the independent variable,
- y is the dependent variable,

the derivative is defined as

$$\begin{aligned} y'(x) = \frac{dy}{dx} &= \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} \\ &= \lim_{h\to 0} \frac{y(x) - y(x-h)}{h}. \end{aligned}$$

The derivative is not defined at points where the "right" and "left" limits do not converge to the same value. For instance, $y(x) = |x|$ does not have a derivative at $x = 0$.

Higher Derivatives

Higher derivatives are defined recursively

$$\begin{aligned} y''(x) &= \frac{d^2y(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy(x)}{dx}\right) \\ y'''(x) &= \frac{d^3y(x)}{dx^3} = \frac{d}{dx}\left(\frac{d^2y(x)}{dx^2}\right) \\ &\text{etc.} \end{aligned}$$

...provided the lower-order derivatives are sufficiently smooth for the higher derivatives to exist.

Notation

- Dash notation:

$$\frac{d}{dx}(.) = (.)'$$
$$\frac{d^2}{dx^2}(.) = (.)''$$
$$\frac{d^n}{dx^n}(.) = (.)^{(n)}$$

- Dot notation: For time-dependent problems, where t is the independent variable, dots are often used to indicate derivatives.

$$x(t)$$
$$\frac{dx}{dt} = \dot{x}(t)$$
$$\frac{d^2x}{dt^2} = \ddot{x}(t)$$

- The dependence on the independent variable may be suppressed. For instance, instead of

$$y'(x) + p(x)\,y(x) = r(x)$$

we can simply write

$$y' + p(x)\,y = r(x)$$

because it's "obvious" that y is a function of x.

Ordinary differential equations

Definition:

- An n-th order ordinary differential equation (ODE) for $y(x)$ has the general form

$$\mathcal{F}\big(x, y(x), y'(x), ..., y^{(n)}(x)\big) = 0, \tag{1}$$

  i.e. it relates the (unknown) function, $y(x)$ to x and its 1st, 2nd, ..., nth derivatives.

- Often the implicit form given above can be solved for $y(x)$, allowing the ODE to be written in explicit form as:

$$y^{(n)}(x) = f\big(x, y(x), y'(x), ..., y^{(n-1)}(x)\big). \tag{2}$$

Solutions:

- A solution of the ODE (1) [or (2)] in an interval

$$I = \{x \mid a < x < b\}$$

  is *any* function $\phi(x)$ for which

$$\mathcal{F}\big(x, \phi(x), \phi'(x), ..., \phi^{(n)}(x)\big) = 0 \qquad \forall x \in I. \tag{3}$$

Notes:

- The statement already suggests that there may be multiple solutions.
- Furthermore, solutions might not exist for all values of x.
- In fact, there might not be a solution at all!

Two properties worth looking out for...

1. Linearity

- An ODE is linear if

$$\mathcal{F}\big(x, y(x), y'(x), ..., y^{(n)}(x)\big) = 0,$$

is linear in y and all its derivatives.

- Linear ODEs can be written as

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y = b(x)$$

where $a_i(x)$ $(i = 0, ..., n)$ and $b(x)$ are given functions.

2. Autonomous ODEs

- An ODE is autonomous if it has the form

$$\mathcal{F}\big(y(x), y'(x), ..., y^{(n)}(x)\big) = 0,$$

i.e. if the independent variable, x, does not appear explicitly.