Algebraic structures in topology

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Overview

I will be talking about A_∞ structures.

They are one example of an algebraic structure arising in topology, with applications to many areas.

They arise when one weakens the notion of associativity to some kind of homotopy associativity.

This example nicely illustrates the interactions between algebra and topology, as well as relations to other areas of mathematics.

A very brief survey

- Topology
  - 1960s [Stasheff]: A_∞-spaces, key example is any loop space ΩX
  - stable homotopy theory, highly structured ring spectra
  - "brave new algebra"

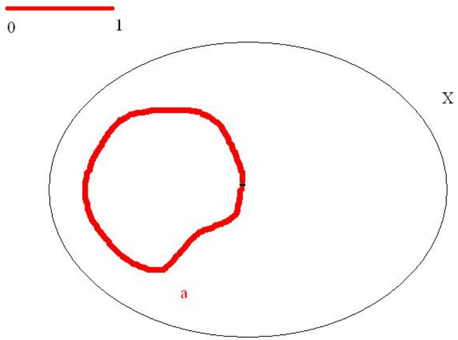
- Algebra
  - A_∞-algebras, key examples are $C_*(\Omega X)$, $\mathrm{Ext}^*_{\mathrm{A}}(\mathrm{M}, \mathrm{M})$
  - study of module categories, derived module categories 1980s [Keller and others]
  - classification results for algebras

A very brief survey, continued

- Mathematical physics
  - A_∞-categories, since 1990s [Fukaya, Kontsevich, ...]
  - key example is Fukaya category of a symplectic manifold
  - related to mirror symmetry
- Also higher category theory, geometry, ...

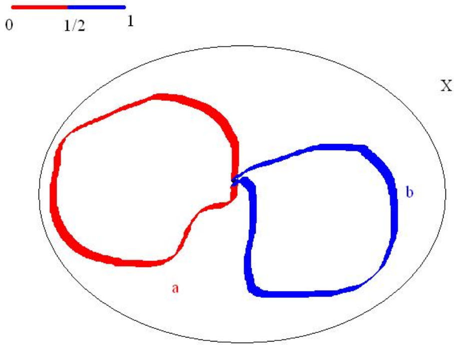
The basic idea

Consider a 'multiplication' which is associative up to homotopy. For example, composition of based loops.
A based loop in a based topological space (X, x_0) is a continuous map $a : [0, 1] \to X$ such that $a(0) = a(1) = x_0$.



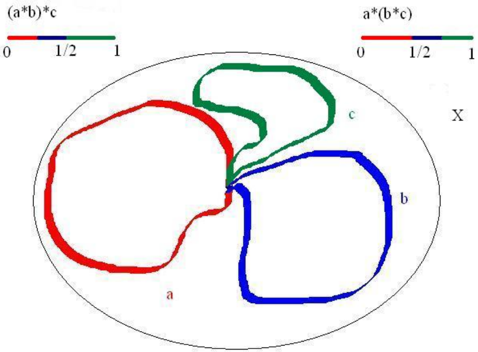
Composition of loops

From two loops, a and b, we obtain a new loop $a * b$ by 'going round a twice as fast and then b twice as fast'.



Homotopy associativity of loop composition

As an immediate consequence of the way we compose loops, we find that composition is not strictly associative, but it is associative up to homotopy.



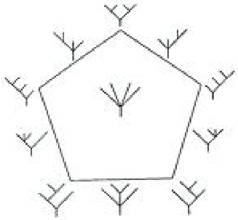
Higher homotopy associativity

- Multiplication

For each pair of points a, b, in $Y = \Omega X$, we have a single point $a * b$.
Multiplication is a map $m_2 : Y \times Y \to Y$.

- Homotopy associativity (K_3)
For each triple of points a, b, c, we have the two points $(a * b) * c$ and $a * (b * c)$ and a path between them.
Thus the (naive) homotopy associativity of the multiplication gives a map $m_3 : Y^3 \times K_3 \to Y$.

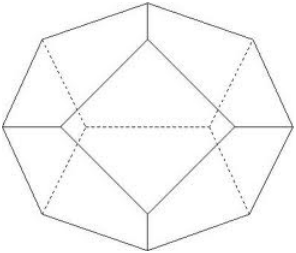
A higher associativity condition (K_4)



Considering 4 points in Y and patching together the information from m_3 we can define a map from the boundary of a pentagon to Y. So we get a map $Y^4 \times \partial K_4 \to Y$. Asking this map to extend over the interior of the pentagon is a higher homotopy associativity condition.

The polytope K_5

There is an inductive procedure which continues this process: each time we can answer ‘yes’ to such a question, a new higher homotopy associativity condition presents itself. The next one involves the figure:



If the answer to all the questions is ‘yes’, we have an A_∞-space. It has a multiplication which is homotopy associative in the strongest possible sense. This is the case for any loop space.

The definition (algebraic)

Definition

An A_∞-algebra structure on a graded k-vector space A is a sequence of k-linear maps $m_j : A^{\otimes j} \to A$ for $j \geq 1$, of degree $j-2$ such that, for each $n \geq 1$,

$$\sum_{i,s} \pm m_{n+1-s}(1 \otimes \cdots \otimes m_s \otimes 1 \otimes \cdots \otimes 1) = 0.$$

Low degrees and special cases

In particular,

- $m_1 : A \to A$ has degree -1 and satisfies $m_1 \circ m_1 = 0$; i.e. it is a differential on A.
- m_1 is a derivation with respect to $m_2 : A^{\otimes 2} \to A$.
- m_3 is chain homotopy associativity of m_2.
- The higher ms encode higher associativity conditions.
- If $m_i = 0$ for all $i \geq 3$, then A is a differential graded associative algebra.
- If $m_1 = 0$, we say A is a minimal A_∞-algebra.
  In this case, the multiplication is strictly associative, but we still have higher ms, encoding lots extra information.

An application: minimal models

Question:
What information about the homology $H_*(A)$ of a differential graded algebra A do you need to recover A, up to quasi-isomorphism?

Answer: [Kadeishvili, Merkulov] (over a field)
An A_∞-algebra structure on $H_*(A)$.

A bit more precisely, $H_*(A)$ admits a unique minimal A_∞-structure in which m_2 is induced by the multiplication on A and such that there is a quasi-isomorphism (of A_∞ algebras) $H_*(A) \to A$.
One can recover A from this structure.

Some recent work

Recently Sagave defined *derived A_∞* algebras in order to have a minimal model theorem which works over a general ground ring.

I have been involved in work to give a more conceptual approach to these structures.

This is an active area of current research.

Topology in the UK

(continued on next page, apologies for any omissions)

- Aberdeen: Richard Hepworth, Ran Levi, Assaf Libman, Jarek Kedra
- Cambridge: Oscar Randal-Williams, Jacob Rasmussen, Burt Totaro
- Durham: Andrew Lobb, Vitaliy Kurlin, Dirk Schuetz,
- Edinburgh: Andrew Ranicki
- Glasgow: Andy Baker
- Kent: Constanze Roitzheim
- Leicester: John Hunton, Frank Neumann, Simona Paoli, Teimuraz Pirashvili

Topology in the UK cont.

- Manchester: Nige Ray, Peter Symonds, Ted Voronov
- Oxford: Christopher Douglas, Marc Lackenby, Graeme Segal, Ulrike Tillmann
- Sheffield: John Greenlees, Neil Strickland, SW, Simon Willerton
- Southampton: Jelena Grbic, Ian Leary, Stephen Theriault
- Swansea: Martin Crossley, Jeff Giansiracusa
- Warwick: John Jones