3.2 Simple Functions

**Definition** A function $f : X \to \mathbb{R}$ is *simple* if it takes only a finite number of different values.

**Note** these values must be finite. Writing them as $a_i, 1 \leq i \leq N$, and letting $A_i = \{x \in X : f(x) = a_i\}$, we can write

$$f = \sum_{i=1}^{N} a_i \chi_{A_i}$$

where $\chi_A$ is the characteristic function of $A$, that is, $\chi_A(x) = 1$ if $x \in A$, and 0 otherwise.

**Lemma 3.7**

The simple functions are closed under addition and multiplication.

**Proof**

Let $s = \sum_{i=1}^{M} a_i \chi_{A_i}$ and $t = \sum_{j=1}^{N} b_j \chi_{B_j}$ where $\bigcup_{i=1}^{M} A_i = \bigcup_{j=1}^{N} B_j = X$.

Define $C_{ij} = A_i \cap B_j$. Then $A_i \subseteq X = \bigcup_{j=1}^{N} B_j$ and so $A_i = A_i \cap \bigcup_{j=1}^{N} B_j = \bigcup_{j=1}^{N} C_{ij}$. Similarly $B_j = \bigcup_{i=1}^{M} C_{ij}$. Since the $C_{ij}$ are disjoint this means that

$$\chi_{A_i} = \sum_{j=1}^{N} \chi_{C_{ij}} \quad \text{and} \quad \chi_{B_j} = \sum_{i=1}^{M} \chi_{C_{ij}}.$$

Thus

$$s = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \chi_{C_{ij}} \quad \text{and} \quad t = \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \chi_{C_{ij}}.$$

Hence

$$s + t = \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) \chi_{C_{ij}} \quad \text{and} \quad st = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i b_j \chi_{C_{ij}}$$

are simple functions. ■

Let $\mathcal{F}$ be a $\sigma$-field on $X$. Assume that for a simple function $f$ we have $A_i \in \mathcal{F}$ for all $i$. Then

$$\{x : f(x) > c\} = \bigcup_{a_i > c} A_i \in \mathcal{F}$$

for all $c \in \mathbb{R}$. Hence $f$ is $\mathcal{F}$-measurable. Conversely assume that $f$ is $\mathcal{F}$-measurable. Order the values attained by $f$ as $a_1 < a_2 < ... < a_N$. Given
1 \leq j \leq N \text{ choose } a_{j-1} < c_1 < a_j < c_2 < a_{j+1}. \text{ (If } j = 1 \text{ or } N \text{ part of this requirement is empty.) Then}

\[
A_j = \left( \bigcup_{a_i > c_1} A_i \right) \setminus \left( \bigcup_{a_i > c_2} A_i \right) = \{ x : f(x) > c_1 \} \setminus \{ x : f(x) > c_2 \} \in \mathcal{F}.
\]

Hence

**Lemma 3.8**

If \( f : (X, \mathcal{F}) \to \mathbb{R} \) is a simple function then \( f \) is \( \mathcal{F} \)-measurable if, and only if, \( A_i \in \mathcal{F} \) for all \( 1 \leq i \leq N \).

**Corollary 3.9**

The simple \( \mathcal{F} \)-measurable functions are closed under addition and multiplication.

**Proof**

Simply note in the proof of Lemma 3.7 that since \( A_i \) and \( B_j \) are in \( \mathcal{F} \) then \( C_{ij} \in \mathcal{F} \).

**Note** If \( s \) is a simple function and \( g : \mathbb{R} \to \mathbb{R} \) is any function whose domain contains the values of \( s \) then \( g \circ s \) (defined by \( (g \circ s)(x) = g(s(x)) \)) is simple. In fact

\[
g \circ s = \sum_{i=1}^{N} g(a_i) \chi_{A_i} = \sum_{j=1}^{M} b_j \chi_{B_i}
\]

for some \( M \leq N \) and where \( B_j = \bigcup_{g(a_i) = b_j} A_i \). Also if \( s \) is \( \mathcal{F} \)-measurable then \( g \circ s \) is too.

The next result is very important.

**Theorem 3.10**

Let \( f \) be a non-negative \( \mathcal{F} \)-measurable function. Then there exist a sequence of simple \( \mathcal{F} \)-measurable functions \( s_n \) such that \( 0 \leq s_1 \leq ... \leq s_n \leq s_{n+1} \leq ... \) and \( \lim_{n \to \infty} s_n = f \).

**Proof**

We partition the range of \( f \) using the points in \( D_n = \{ \frac{\nu}{2^n} : 0 \leq \nu \leq n2^n \} \).

Importantly, though trivial, we have \( D_n \subseteq D_{n+1} \).

Define \( s_n(x) = \max\{ \gamma \in D_n, \gamma \leq f(x) \} \).

Then \( D_n \subseteq D_{n+1} \) means that for any given \( x \),
Amadeus was a letter from the sky. He was a shepherd. He was a wizard. He was a king.
Combining Theorems 3.10 and 3.6 we see that a function \( f : (X, \mathcal{F}) \to \mathbb{R}^+ \) is \( \mathcal{F} \)-measurable if, and only if, there exists an increasing sequence of simple, \( \mathcal{F} \)-measurable functions converging to \( f \).

**Corollary 3.11**

If \( f : (X, \mathcal{F}) \to \mathbb{R}^+ \) is \( \mathcal{F} \)-measurable then it is the limit of a sequence of simple \( \mathcal{F} \)-measurable functions.

**Proof**

As in the proof of Theorem 3.4(viii) we can write \( f = f^+ - f^- \) where \( f^+ \) and \( f^- \) are non-negative \( \mathcal{F} \)-measurable functions. So by Theorem 3.10 we can find sequences of simple, \( \mathcal{F} \)-measurable functions \( s_n \to f^+ \) and \( t_n \to f^- \) in which case \( \{s_n - t_n\}_{n \geq 1} \) is the required sequence of simple functions (using Lemma 3.7) converging to \( f \). \[ \blacksquare \]

**Corollary 3.12**

If \( f : (X, \mathcal{F}) \to \mathbb{R}^* \) is \( \mathcal{F} \)-measurable and \( g : \mathbb{R} \to \mathbb{R} \) a continuous function whose domain contains the values of \( f \) then the composition function \( g \circ f \) is \( \mathcal{F} \)-measurable.

**Proof**

By Corollary 3.11 we can find a sequence of simple, \( \mathcal{F} \)-measurable functions \( s_n \to f \). By an earlier note the functions \( g \circ s_n \) are simple and still \( \mathcal{F} \)-measurable for all \( n \). Then

\[
\lim_{n \to \infty} g(s_n(x)) = g(\lim_{n \to \infty} s_n(x)) \quad \text{since } g \text{ is continuous,}
\]

\[
= g(f(x))
\]

\[
= (g \circ f)(x)
\]

for all \( x \in X \), i.e. \( g \circ f = \lim_{n \to \infty} g \circ s_n \). Hence, by Theorem 3.6, \( g \circ f \) is \( \mathcal{F} \)-measurable. \[ \blacksquare \]

**Example 12** If \( f : (X, \mathcal{F}) \to \mathbb{R}^+ \) is \( \mathcal{F} \)-measurable then \( \sin f \), \( \exp(f) \) and \( \log f \) are also \( \mathcal{F} \)-measurable on the set of \( x \) on which they are defined.