Part 4. Integration (with Proofs)

4.1 Definition

**Definition** A partition $\mathcal{P}$ of $[a, b]$ is a finite set of points $\{x_0, x_1, ..., x_n\}$ with $a = x_0 < x_1 < ... < x_n = b$.

**Example** $\{0, \frac{1}{4}, \frac{1}{3}, \frac{7}{8}, 1\}$ and $\{0, \frac{1}{100}, \frac{2}{100}, 1\}$ are partitions of $[0, 1]$.

For each $n \geq 1$ define $P_n = \left\{a + \left(\frac{b-a}{n}\right)i : 1 \leq i \leq n\right\}$, the arithmetic partition of $[a, b]$.

Assume $b > a > 0$. For each $n \geq 1$ define $Q_n = \left\{a \left(\frac{b}{a}\right)^{i/n} : 1 \leq i \leq n\right\}$, the geometric partition of $[a, b]$.

A partition of $[a, b]$ divides the interval into $n$ sub-intervals $[x_{i-1}, x_i], 1 \leq i \leq n$. Let $f$ be a bounded function on $[a, b]$, so there exist $m$ and $M$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. For each $1 \leq i \leq n$ let

$$M_i = \text{lub} \{f(x) : x \in [x_{i-1}, x_i]\},$$

and

$$m_i = \text{glb} \{f(x) : x \in [x_{i-1}, x_i]\},$$

which exist by the Completeness of $\mathbb{R}$. So $m \leq m_i \leq M_i \leq M$ for all $1 \leq i \leq n$.

**Definition** For $f$ bounded on $[a, b]$ the Upper Sum is

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

and the Lower Sum is

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}).$$
Example An upper sum

\[
\begin{array}{c}
\text{A lower sum}
\end{array}
\]

It would appear from these diagrams that the lower and upper sums are the areas of regions which either contain or are contained within the region under the graph between \(a\) and \(b\). Be careful, I have drawn a “nice, smooth” function, the situation might look different with more “pathological” functions. Also be careful because I have drawn only non-negative functions. Look at what happens if the function should be negative for some \(x\). If we could assign a measure of size to the region under a graph we might expect it to be less than the areas measured by \(U(P, f)\) for all \(P\), yet greater than those measured by \(L(P, f)\) for all \(P\).

Example Let \(f : [0, 1] \to \mathbb{R}\) be given by \(f(x) = x - x^2\). Find \(U(P, f)\) and \(L(P, f)\) when

(i) \(P = \left\{ 0, \frac{1}{2}, 1 \right\}\) and (ii) \(P = \left\{ 0, \frac{1}{3}, \frac{2}{3}, 1 \right\}\).

Example Let \(f : [1, 2] \to \mathbb{R}\) be given by \(f(x) = x\). Find \(U(P_n, f)\), \(L(P_n, f)\), \(U(Q_n, f)\) and \(L(Q_n, f)\) for the arithmetic and geometric partitions.

Lemma For all partitions \(P\) of \([a, b]\) we have
\[ m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a). \]

**Proof** Left to student.

From this we see that for \( f \) bounded on \([a, b]\),

\[ \{U(P, f) : \mathcal{P} \text{ a partition of } [a, b]\} \]

is a non-empty set of real numbers bounded below, by \( m(b - a) \). So by Completeness of the Reals this set has a greatest lower bound. Similarly,

\[ \{L(P, f) : \mathcal{P} \text{ a partition of } [a, b]\} \]

has a least upper bound.
**Definition** For \( f \) bounded on \([a, b]\), the *Upper Integral* is

\[
\int_{a}^{b} f = \text{glb} \{ U(P, f) : \text{\( P \) a partition of \([a, b]\)} \}
\]

(1)

and the *Lower Integral* is

\[
\int_{a}^{b} f = \text{lub} \{ L(P, f) : \text{\( P \) a partition of \([a, b]\)} \}.
\]

**Definition** If \( P \) and \( D \) are two partitions of a set which satisfy \( P \subseteq D \), we say that \( D \) is a *refinement* of \( P \). (We also say that \( D \) is *finer* than \( P \) and, equivalently, \( P \) is *coarser* than \( D \).)

**Lemma** If \( f \) is bounded on \([a, b]\) and \( D \) is a *refinement* of \( P \) then

\[
L(P, f) \leq L(D, f) \leq U(D, f) \leq U(P, f).
\]

**Corollary** If \( f \) is any bounded function on \([a, b]\) then

\[
\int_{a}^{b} f \leq \int_{a}^{b} f.
\]

**Definition** A bounded function \( f \) on \([a, b]\) is *Riemann integrable over* \([a, b]\) if

\[
\int_{a}^{b} f = \int_{a}^{b} f.
\]

The common value is called the (Riemann) integral and is denoted by \( \int_{a}^{b} f \) or \( \int_{a}^{b} f(x) \, dx \).

**Note** To save time in lectures I will often write \( \int \)ation and \( \int \)able in place of integration and integrable.
Riemann gave his definition of an integral in 1854. Here we have given an approach due to Darboux from 1875. The upper and lower sums above should strictly be called Upper and Lower Darboux sums. They differ slightly from the Upper and Lower Riemann sums that you might find in alternative accounts of integration. But be careful! In a book by Strichartz the Darboux sums are called Riemann sums while, what I would call Riemann sums, are called Cauchy sums. Very confusing! (♣)

**Example** Let $f : [0, 1] \to \mathbb{R}, x \mapsto x$. Prove, by verifying the definition, that $f$ is Riemann integrable over $[0, 1]$. What is the value of the integral?

**Hint:** Use a sequence of Arithmetic Partitions.

**Example** Let $f : [1, 2] \to \mathbb{R}, x \mapsto \frac{1}{x}$. Prove, by verifying the definition, that $f$ is Riemann integrable over $[1, 2]$. What is the value of the integral?

**Hint:** Use a sequence of Geometric Partitions.

We now return to an example seen earlier in Question 1a.2.

**Example** (i) Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

Show that $f$ is **not** Riemann integrable over $[0, 1]$.

(♣) Though earlier in the course I termed this function pathological it is not, in the scheme of things, a complicated function (imagine what a continuous nowhere-differentiable function might look like). So it is a weakness of the theory of Riemann integration that we can’t integrate this function. In course 341 a theory of integration due to Lebesgue, from 1902, is studied. With Lebesgue integration this function can be integrated. Can you guess at the value of the integral? (♣)

(ii) Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1/q & \text{if } x \in \mathbb{Q} \text{ is written as } p/q \text{ in reduced form,} \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

(This was seen in Question 2.17(ii), and is discontinuous at every rational.)

Show that $f$ **is** Riemann integrable over $[0, 1]$. Question ??♣
4.2 Criteria for being Riemann Integrable.

**Lemma** If \( f : [a, b] \to \mathbb{R} \) is Riemann integrable then,

\[
\forall \varepsilon > 0, \exists \mathcal{P} : \int_a^b f - \varepsilon < L(\mathcal{P}, f) \leq U(\mathcal{P}, f) < \int_a^b f + \varepsilon.
\]

**Proof** \( f \) integrable means

\[
\int_a^b f = \int_a^b f = \int_a^b f. \tag{2}
\]

Let \( \varepsilon > 0 \) be given. Then \( \int_a^b f = \text{glb}\{U(\mathcal{P}, f)\}_{\mathcal{P}} \) means there exists \( \mathcal{P}_1 \) such that

\[
U(\mathcal{P}_1, f) < \int_a^b f + \varepsilon = \int_a^b f + \varepsilon \tag{3}
\]

by (2). Similarly, \( \int_a^b f = \text{lub}\{L(\mathcal{P}, f)\}_{\mathcal{P}} \) means there exists \( \mathcal{P}_2 \) such that

\[
L(\mathcal{P}_2, f) > \int_a^b f + \varepsilon = \int_a^b f + \varepsilon \tag{4}
\]

by (2). Set \( \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \). Then (3) and (4) hold with \( \mathcal{P} \) in place of \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), which is the statement of this lemma.

The next result is of great use when proving properties of integration. It is not so important when showing a given function is integrable.

**Theorem** Riemann’s Criteria.

The bounded function \( f : [a, b] \to \mathbb{R} \) is Riemann integrable if, and only if,

\[
\forall \varepsilon > 0, \exists \mathcal{P} : U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon. \tag{5}
\]

**Proof** (\( \iff \)) Assume (5) holds. From

\[
L(\mathcal{P}, f) \leq \int_a^b f \leq \int_a^b f \leq U(\mathcal{P}, f)
\]

we see that

\[
0 \leq \int_a^b f - \int_a^b f < \varepsilon.
\]
True for all $\varepsilon > 0$ means we must have equality throughout, and so $f$ is Riemann integrable.

$(\Rightarrow)$ Assume that $f$ is Riemann integrable. Let $\varepsilon > 0$ be given. From Lemma (with $\varepsilon/2$ in place $\varepsilon$) we can find $P$ such that

$$\int_a^b f - \varepsilon/2 < L(P, f) \leq U(P, f) < \int_a^b f + \varepsilon/2,$$

i.e. $U(P, f) - L(P, f) < \varepsilon$ as required.

Theorem If $f : [a, b] \to \mathbb{R}$ is monotonic then $f$ is Riemann integrable.

Theorem If $f : [a, b] \to \mathbb{R}$ is continuous then $f$ is Riemann integrable.

4.3 Integration Rules

For $A, B \subseteq \mathbb{R}$ define $A + B = \{a + b : a \in A, b \in B\}$, the sumset of $A$ and $B$. Further, given $A \subseteq \mathbb{R}$ write $-A = \{-a : a \in A\}$ then the sumset $A - A = A + (-A) = \{a - a' : a, a' \in A\}$. (Not the empty set as might have been thought!)

Lemma If $A, B \subseteq \mathbb{R}$ then

$$\operatorname{lub}(A + B) = \operatorname{lub}A + \operatorname{lub}B \quad \text{and} \quad \operatorname{glb}(A + B) = \operatorname{glb}A + \operatorname{glb}B$$

Proof For all $a \in A$ we have $a \leq \operatorname{lub}A$, and for all $b \in B$ we have $b \leq \operatorname{lub}B$. Adding we get

$$a + b \leq \operatorname{lub}A + \operatorname{lub}B \quad \text{for all} \quad a + b \in A + B.$$ 

(So $\operatorname{lub}A + \operatorname{lub}B$ is an upper bound for $A + B$ for which $\operatorname{lub}(A + B)$ is the least of all upper bounds.) Hence

$$\operatorname{lub}(A + B) \leq \operatorname{lub}A + \operatorname{lub}B. \quad (6)$$

Conversely, $a + b \in A + B$ means that $a + b \leq \operatorname{lub}(A + B)$ for all $a + b \in A + B$, i.e. for all $a \in A$ and all $b \in B$.

Let $b \in B$ be given. So we have $a \leq \operatorname{lub}(A + B) - b$ for all $a \in A$. That is, $\operatorname{lub}(A + B) - b$ is an upper bound for $A$ while $\operatorname{lub}A$ is the least of all such upper bounds. Hence $\operatorname{lub}A \leq \operatorname{lub}(A + B) - b$.  

7
We thus have \( b \leq \text{lub} \,(A + B) - \text{lub} \,A \) for all \( b \in B \). As in the last paragraph this means that \( \text{lub} \,B \leq \text{lub} \,(A + B) - \text{lub} \,A \), or

\[
\text{lub} \,A + \text{lub} \,B \leq \text{lub} \,(A + B) . \tag{7}
\]

Combine (6) and (7) to get result.

I leave the proof for glb to student. ■

**Application**

The sumset \( A + B = A - A = \{a - a' : a, a' \in A\} \) has the property that if \( a - a' \in A - A \) then \( a' - a \in A - A \). Alternatively \( \eta \in A - A \) iff \( -\eta \in A - A \), so we could write

\[
A - A = \{|\eta| : \eta \in A - A\} \cup \{-|\eta| : \eta \in A - A\}
\]

Strange perhaps, but it does allow us to say that

\[
\text{lub} \,(A - A) = \text{lub} \,\{|\eta| : \eta \in A - A\} . \tag{8}
\]

Finally note that \( \text{lub} \,B = \text{lub} \,(-A) = -\text{glb} \,(A) \).

So the Lemma gives

\[
\text{lub} \,A - \text{glb} \,A = \text{lub} \,A + \text{lub} \,(-A) = \text{lub} \,(A + B) \\
= \text{lub} \,\{|a - a' : a, a' \in A\}\, ,
\]

by (8). Apply this with \( A_i = \{f(x) : x \in [x_{i-1}, x_i]\} \) to deduce

\[
M_i^f - m_i^f = \text{lub} \,\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}
\]

This is useful when combined with Riemann’s Criteria.

**Theorem** Assume that the bounded functions \( f \) and \( g \) are Riemann integrable on \([a, b]\). Then

(i) Sum Rule: \( f + g \) is integrable on \([a, b]\).

(ii) Product Rule: \( fg \) is integrable on \([a, b]\).

(iii) Quotient Rule: \( f/g \) is integrable on \([a, b]\) if there exists \( C > 0 \) such that \( |g(x)| \geq C \) for all \( x \in [a, b] \).

(iv) \(|f|\), defined by \(|f|(x) = |f(x)|\) for all \( x \in [a, b] \), is integrable over \([a, b]\).

**Proof** (i) For \( x \in [x_{i-1}, x_i] \) we have \( m_i^f \leq f(x) \) and \( m_i^g \leq g(x) \), and so \( m_i^f + \)
\[ m_i^g \leq f(x) + g(x). \] Thus \( m_i^f + m_i^g \) is a lower bound for \( \{ f(x) + g(x) : x \in [x_{i-1}, x_i] \} \), for which \( m_i^{f+g} \) is the greatest lower bound. Hence
\[ m_i^f + m_i^g \leq m_i^{f+g}. \]

Similarly
\[ M_i^f + M_i^g \geq M_i^{f+g}. \]

From these we get
\[
U(P, f + g) = U(P, f) + U(P, g),
\]
and
\[
L(P, f + g) = L(P, f) + L(P, g).
\]

Hence
\[
U(P, f + g) - L(P, f + g) \leq U(P, f) - L(P, f) + U(P, g) - L(P, g),
\]
for all partitions \( P \).

Let \( \varepsilon > 0 \) be given. The fact that \( f \) is integrable implies, by Riemann’s Criteria, that we can find \( P_1 \) such that
\[
U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2}.
\]
For the same reasons we can find \( P_2 \) such that
\[
U(P_2, g) - L(P_2, g) < \frac{\varepsilon}{2}.
\]
Take \( P_0 = P_1 \cup P_2 \). Then (12) and (13) both hold for this partition. From (11) we then get
\[
U(P_0, f + g) - L(P_0, f + g) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus, by Riemann’s Criteria, we find that \( f + g \) is Riemann integrable over \([a, b]\).

(ii) Let \( K > 0 \) and \( N > 0 \) be bounds on \( f \) and \( g \), so \( |f(x)| \leq K \) and \( |g(x)| \leq N \) for all \( x \in [a, b] \).

We use Application and examine
\[ M_{fg}^i - m_{fg}^i = \text{lub} \{|f(x)g(x) - f(y)g(y)| : x, y \in [x_{i-1}, x_i] \}. \]

In this we use ideas seen when examining limits of products,

\[
|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|
\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|
\leq K|g(x) - g(y)| + N|f(x) - f(y)|.
\]

So we get

\[ M_{fg}^i - m_{fg}^i \leq K \left( M_i^f - m_i^f \right) + N \left( M_i^g - m_i^g \right). \]

Thus

\[
U(\mathcal{P}, fg) - L(\mathcal{P}, fg) \leq K \left( U(\mathcal{P}, f) - L(\mathcal{P}, f) \right) + N \left( U(\mathcal{P}, g) - L(\mathcal{P}, g) \right)
\]

for all partitions \( \mathcal{P} \), again by the Application. Again finish the proof as in (i) using Riemann’s criteria on \( f \) with \( \varepsilon/2K \), on \( g \) with \( \varepsilon/2N \) and then combining the two partitions found.

(iii) It suffices to prove this result for \( 1/g \); that for \( f/g \) follows then by part (ii). From the Application,

\[ M_{1/g}^i - m_{1/g}^i = \text{lub} \left\{ \left| \frac{1}{g(x)} - \frac{1}{g(y)} \right| : x, y \in [x_{i-1}, x_i] \right\}. \]

In this simply use

\[
\left| \frac{1}{g(x)} - \frac{1}{g(y)} \right| \leq \frac{|g(x) - g(y)|}{C^2}
\]

to get

\[ M_{1/g}^i - m_{1/g}^i \leq \frac{1}{C^2} \left( M_i^g - m_i^g \right), \]

and

\[
U\left(\mathcal{P}, \frac{1}{g}\right) - L\left(\mathcal{P}, \frac{1}{g}\right) \leq \frac{1}{C^2} \left( U(\mathcal{P}, g) - L(\mathcal{P}, g) \right)
\]
for all partitions $\mathcal{P}$. So finish off with Riemann’s Criteria with $C^2\varepsilon$.

(iv) For $a, b \in \mathbb{R}$ we have $|a| = |a - b + b| \leq |a - b| + |b|$ and

$$|b| = |-b| = |a - b - a| \leq |a - b| + |-a| = |a - b| + |a|.$$ 

So $-|a - b| \leq |a| - |b| \leq |a - b|$ and thus $||a| - |b|| \leq |a - b|$.

Using the Application we deduce

$$M^{|f|}_i - m^{|f|}_i \leq M^f_i - m^f_i,$$

and

$$U(\mathcal{P}, |f|) - L(\mathcal{P}, |f|) \leq U(\mathcal{P}, f) - L(\mathcal{P}, f)$$

for all partitions $\mathcal{P}$. Then an application of Riemann’s Criteria will give $|f|$ integrable.

**Theorem** Assume that the bounded functions $f$ and $g$ are Riemann integrable on $[a, b]$. Then

(i) Linearity.

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$ 

(ii) Additive Property. For $a < c < b$,

$$\int_a^b f = \int_a^c f + \int_c^b f.$$ 

(iii) If $f(x) \leq g(x)$ for all $x \in [a, b]$ then

$$\int_a^b f \leq \int_a^b g.$$ 

(iv)

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$ 

**Note** It fits in with these results if we define

$$\int_a^b f(t) \, dt = -\int_b^a f(t) \, dt.$$

**Proof** (i) From (9) and (10) we have
$L(P, f) + L(P, g) \leq L(P, f + g) \leq U(P, f + g) \leq U(P, f) + U(P, g).$ \hfill (14)

Let $\varepsilon > 0$ be given. From the Lemma above we can find $P^f$ and $P^g$ such that

$$\int_a^b f - \frac{\varepsilon}{2} \leq L(P^f, f) \leq \int_a^b f + \frac{\varepsilon}{2} \quad \hfill (15)$$

and

$$\int_a^b g - \frac{\varepsilon}{2} \leq L(P^g, g) \leq \int_a^b g + \frac{\varepsilon}{2}. \quad \hfill (16)$$

Let $P = P^f \cup P^g$, when both (15) and (16) hold for this partition.

From (14), (15), (16) and the fact that $f + g$ is integrable, we find that

$$\int_a^b f + \int_a^b g - \varepsilon \leq L(P, f + g) \leq \int_a^b (f + g) \leq U(P, f + g) \leq \int_a^b f + \int_a^b g + \varepsilon.$$

True for all $\varepsilon > 0$ gives the result.

(ii) I leave it to the student to check that if $f$ is integrable over an interval $I$ then it is integrable over any subinterval $J \subseteq I$.

So, since $f$ is now integrable over $[a, c]$, there exists $P^a$, a partition of $[a, c]$, such that

$$\int_a^c f - \frac{\varepsilon}{2} \leq L(P^a, f) \leq \int_a^c f + \frac{\varepsilon}{2}. \quad \hfill (17)$$

Similarly, there exists $P^b$, a partition of $[c, b]$, such that

$$\int_c^b f - \frac{\varepsilon}{2} \leq L(P^b, f) \leq \int_c^b f + \frac{\varepsilon}{2}. \quad \hfill (18)$$

Set $P = P^a \cup P^b$, a partition of $[a, b]$. Then

$$U(P, f) = U(P^a, f) + U(P^b, f) \quad \hfill (19)$$

$$L(P, f) = L(P^a, f) + L(P^b, f). \quad \hfill (20)$$

Combining (17) – (20) gives
\[
\int_a^c f + \int_c^b f - \varepsilon \leq \underline{L}(\mathcal{P}, f) \leq \int_a^b f \leq \overline{L}(\mathcal{P}, f) \leq \int_a^c f + \int_c^b f + \varepsilon.
\]

True for all \( \varepsilon > 0 \) means we have equality in the centre, so \( f \) is integrable over \([a, b]\) with value \( \int_a^c f + \int_c^b f \).

(iii) Let \( \varepsilon > 0 \) be given. Since \( g \) is integrable we can find \( \mathcal{P} \) such that
\[
\int_a^b g - \varepsilon \leq \underline{L}(\mathcal{P}, g).
\]
The assumption that \( g \leq f \) means that \( \underline{L}(\mathcal{P}, g) \leq \underline{L}(\mathcal{P}, f) \). Finally by definition, \( \underline{L}(\mathcal{P}, f) \leq \int_a^b f \). Combing these three facts gives
\[
\int_a^b g - \varepsilon \leq \underline{L}(\mathcal{P}, g) \leq \underline{L}(\mathcal{P}, f) \leq \int_a^b f.
\]
True for all \( \varepsilon > 0 \) means that \( \int_a^b g \leq \int_a^b f \).

(iii) We have seen earlier that \(|f|\) is integrable. Use part (ii) on \(-|f| \leq f \leq |f|\) to get the inequality. \(\blacksquare\)

Though not needed for our course, a useful alternative form of Riemann’s Criteria is

♣ Theorem A bounded function \( f : [a, b] \to \mathbb{R} \) is Riemann integrable iff there exists a sequence of partitions \( \{\mathcal{P}_n\}_{n \geq 1} \) of \([a, b]\) with \( \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \subseteq \ldots \) such that \( \lim_{n \to \infty} \overline{U}(\mathcal{P}_n, f) = \lim_{n \to \infty} \underline{L}(\mathcal{P}_n, f) \). This common value is the value of the integral.

Proof \((\Leftarrow)\) Simply use
\[
\underline{L}(\mathcal{P}_n, f) \leq \int_a^b f \leq \overline{L}(\mathcal{P}_n, f)
\]
for all \( n \geq 1 \), along with the Sandwich Theorem.

\((\Rightarrow)\) Assume that \( f \) is Riemann integrable in which case we have Riemann’s Criteria. Let \( n \geq 1 \) be given. Choose \( \varepsilon = 1/n \) in Riemann’s Criteria to find a partition \( \mathcal{Q}_n : \overline{U}(\mathcal{Q}_n, f) - \underline{L}(\mathcal{Q}_n, f) < 1/n \).

The \( \{\mathcal{Q}_n\} \) may well not satisfy \( \mathcal{Q}_1 \subseteq \mathcal{Q}_2 \subseteq \mathcal{Q}_3 \subseteq \ldots \) so define
\[ \mathcal{P}_n = \bigcup_{1 \leq j \leq n} Q_j. \]

Then
\[ \mathcal{P}_n = \bigcup_{1 \leq j \leq n} Q_j \subseteq \bigcup_{1 \leq j \leq n+1} Q_j = \mathcal{P}_{n+1}. \]

And \( Q_n \subseteq \mathcal{P}_n \) means \( U(\mathcal{P}_n, f) \leq U(Q_n, f) \) and \( L(\mathcal{P}_n, f) \geq L(Q_n, f) \) so
\[ U(\mathcal{P}_n, f) - L(\mathcal{P}_n, f) \leq U(Q_n, f) - L(Q_n, f) < 1/n. \]

Next, \( \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \subseteq ... \) means that \( U(\mathcal{P}_1, f) \geq U(\mathcal{P}_2, f) \geq U(\mathcal{P}_3, f) \geq ... \), a decreasing sequence bounded below by \( m(b-a) \). Thus it converges, to \( \alpha \) say.

Similarly, \( L(\mathcal{P}_1, f) \leq L(\mathcal{P}_2, f) \leq L(\mathcal{P}_3, f) \leq ... \), an increasing sequence bounded above by \( M(b-a) \). Thus this also converges, to \( \beta \) say.

Finally,
\[ 0 \leq \alpha - \beta = \lim_{n \to \infty} (U(\mathcal{P}_n, f) - L(\mathcal{P}_n, f)) \leq \lim_{n \to \infty} 1/n = 0. \]

Hence \( \alpha = \beta \) and so \( \lim_{n \to \infty} U(\mathcal{P}_n, f) = \lim_{n \to \infty} L(\mathcal{P}_n, f) \) while
\[ L(\mathcal{P}_n, f) \leq \int_a^b f \leq U(\mathcal{P}_n, f) \]
gives the value of the limit as \( \int_a^b f \).

\[ \square \]

\( \text{♣} \) For the statement of our next criteria define \( \partial\mathcal{P} \), of a partition \( \mathcal{P} = \{a = x_0 < x_1 < ... < x_n = b\} \), to be \( \partial\mathcal{P} = \max_{1 \leq i \leq n} |x_i - x_{i-1}| \), i.e. the maximum partition length.

**Theorem** (Du Bois-Reymond 1875, Darboux 1875) A bounded function \( f : [a, b] \to \mathbb{R} \) is Riemann Integrable if, and only if,
\[ \forall \varepsilon > 0, \exists \delta > 0, \forall \mathcal{P} \in D_\delta, U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon. \]

Here \( D_\delta \) is the set of all partitions with \( \partial\mathcal{P} < \delta \). (\( \text{♣} \))

**Example** Use the arithmetic partitions along with Riemann’s criteria to show that \( f : [0, 3] \to \mathbb{R} : x \mapsto x \) is integrable and find the value of \( \int_0^3 t \, dt \).

**Theorem** If \( f \) is continuous on \([a, b]\) then \( f \) is Riemann integrable on \([a, b]\).
Proof Not given in this course, but if interested, see

http://www.ma.umist.ac.uk/mdc/211/proof.pdf

(♣) The proof depends on a property of continuous functions that we have not discussed and which you will not study until course 251. This property is that if \( f: [a, b] \rightarrow \mathbb{R} \) is continuous then, since the domain \([a, b]\) is closed and bounded, we have

\[
\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in [a, b], |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon. \quad (21)
\]

Try to understand why this statement is different to the definition of continuity. A function that satisfies (21) is said to be uniform continuity. (♣)

See Question ??

Theorem If \( f \) is monotonic on \([a, b]\) then \( f \) is Riemann integrable on \([a, b]\).

Example Let \( f: [0, 1] \rightarrow \mathbb{R} \) be given by \( f(x) = \frac{1}{1 + x^2} \). Prove that \( f \) is Riemann integrable over \([0, 1]\).

Example Let \( f: [0, 1] \rightarrow \mathbb{R} \) be given by \( f(0) = 0 \) and, for \( x \in (0, 1] \),

\[
f(x) = \frac{1}{n} \text{ where } n \text{ is the largest integer satisfying } x \leq \frac{1}{n}.
\]

Draw the graph of \( f \). Show that \( f \) is Riemann integrable on \([0, 1]\). How many discontinuities does this function have? Question ??
4.4 Integration as the inverse of differentiation.

**Definition** Let \( f : (a, b) \rightarrow \mathbb{R} \) be given. Assume there exists a function \( F : [a, b] \rightarrow \mathbb{R} \) which is continuous on \([a, b]\), differentiable on \((a, b)\) and satisfies \( F'(x) = f(x) \) for all \( x \in (a, b) \). Then \( F \) is called a **primitive** (or **anti-derivative**) of \( f \).

**Note** If a primitive exists it will not be unique. If \( F \) is a primitive for \( f \) then \( F + c, \) where \( c \) is a constant, satisfies \( (F + c)' = f \). Thus \( F + c \) is also a primitive for \( f \). But if \( F_1 \) and \( F_2 \) are two primitives for \( f \) then \( F_1' = f = F_2' \). Thus \( (F_1 - F_2)' = 0 \). By the Increasing-Decreasing Theorem this means that \( F_1 - F_2 = c \), for some constant \( c \). So, if \( F \) is a primitive for \( f \) then \( F_0 \) is a primitive if, and only if, \( F_0 - F \) is constant on \([a, b]\).

**Theorem** Fundamental Theorem of Calculus.

1) If \( f \) is Riemann Integrable on \([a, b]\) then
\[
F(x) = \int_a^x f(t) \, dt
\]

is continuous on \([a, b]\).

2) Further, if \( f \) is continuous on \([a, b]\) then \( F \) is differentiable on \((a, b)\) and \( F'(x) = f(x) \) for all \( x \in (a, b) \).

**Proof** 1) Since \( f \) is bounded choose \( N > 0 : |f(x)| \leq N \) for all \( x \in [a, b] \).

Let \( \varepsilon > 0 \) be given. Let \( c \in (a, b) \) be given.

Choose \( \delta = \varepsilon / N \).

Assume \(|x - c| < \delta \). If \( x > c \) consider

\[
|F(x) - F(c)| = \left| \int_c^x f(t) \, dt \right| = \left| \int_c^x f(t) \, dt \right| \\
\leq \int_c^x |f(t)| \, dt \leq \int_c^x N \, dt = N(x - c).
\]

If \( c < x \) consider \(|F(x) - F(c)| = |F(c) - F(x)| \leq N(c - x) \) by above.

So, in all cases,
\[
|F(x) - F(c)| \leq N|x - c| < N\delta = \varepsilon.
\]
Hence $F$ is continuous at $c$ and thus on $(a, b)$.

2) Let $\varepsilon > 0$ be given. Let $c \in (a, b)$ be given.

We are told that $f$ is continuous at $c$ so there exists $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.

Assume $|x - c| < \delta$. If $x > c$ consider

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \frac{1}{|x - c|} \left| (F(x) - F(c)) - (x - c) f(c) \right|$$

$$= \frac{1}{|x - c|} \left| \int_c^x f(t) dt - \int_c^x f(c) dt \right|$$

$$\leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)| dt$$

$$\leq \frac{1}{|x - c|} \int_c^x \varepsilon dt = \varepsilon.$$

Here we have used $|t - c| \leq |x - c| < \delta$ and so $|f(t) - f(c)| < \varepsilon$ inside the integral. If $x < c$ the same result follows on writing

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \left| \frac{F(c) - F(x)}{c - x} - f(c) \right|.$$ 

So in all cases

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| < \varepsilon.$$

Thus

$$\lim_{x \to c} \frac{F(x) - F(c)}{x - c}$$

eexists, i.e. $F$ is differentiable at $c$, with derivative $f(c)$.

**Note:** If $f$ is integrable, this is not sufficient to say that $F(x) = \int_a^x f(t) dt$ is a primitive for $f$. We need the extra condition that $f$ is continuous.

**Corollary** If $g$ is continuous on $[a, b]$ and $G$ is a primitive for $g$ then

$$\int_y^x g(t) dt = G(x) - G(y) = [G(t)]_y^x$$

for all $a \leq y < x \leq b$.

**Example** Prove that $\ln x$, defined earlier as the inverse of $e^x$, satisfies
\[ \ln x = \int_1^x \frac{dt}{t} \]

for all \( x > 0 \). This is often taken as the definition of the natural logarithm.

**Question** ??

**Example** (Cauchy) Prove that if \( f^{(n+1)} \) is continuous on \([a,b]\) then

\[ \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) \, dt = R_{n,a} f(x) \]

for all \( a \leq x \leq b \), where \( R_{n,a} \) is the remainder term for the \( n \)-th Taylor Series.

**Proof** Let \( x \in [a,b] \). By an earlier example we have seen that the Taylor polynomial, \( T_{n,t} f(x) \) is a primitive for

\[ \frac{(x-t)^n}{n!} f^{(n+1)}(t), \]

as a function of \( t \). Apply the corollary with \( g(t) = \frac{(x-t)^n}{n!} f^{(n+1)}(t) \), continuous by assumption on \( f \), to get

\[ \int_c^d \frac{(x-t)^n}{n!} f^{(n+1)}(t) \, dt = T_{n,d} f(x) - T_{n,c} f(x), \]

for all \( c,d \in [a,b] \). Take \( c = a, d = x \), and recall \( T_{n,x} f(x) = f(x) \). Thus we get our result. \( \blacksquare \)

**Applications** We have

\[ \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots = \sum_{r=1}^{\infty} \frac{(-1)^{r-1} x^r}{r} \]

valid for \(-1 < x \leq 1\).

**Note** In particular \( \lim_{n \to \infty} R_{n,0} (\ln (1 + x))|_{x=1} = 0 \). Putting \( x = 1 \) gives

\[ \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots. \]

The Fundamental Theorem tells us how to differentiate an integral.

**Example** Let

\[ G(x) = \int_{x^2}^{x^3} e^t \cos t \, dt. \]

Calculate \( G'(t) \).
Solution. Let

\[ F(y) = \int_1^y e^t \cos t \, dt, \]

so \( G(x) = F(x^3) - F(x^2) \). Since \( e^t \cos t \) is continuous we have, by the Fundamental Theorem, that \( F \) is differentiable. Thus

\[ G'(x) = 3x^2 F'(x^3) - 2xF'(x^2) = 3x^2 e^{x^3} \cos (x^3) - 2xe^{x^2} \cos (x^2). \]

As a further application, note that the rule for differentiating a product of functions \( g \) and \( h \) can be restated as saying that \( gh \) is a primitive for \( gh' + g'h \). So, if \( gh' + g'h \) is continuous, we can apply the Fundamental Theorem, giving

\[ \int_y^x (gh' + g'h) (t) \, dt = [(gh) (t)]_y^x. \]

Thus we have justified Theorem Integration by parts.

Assume that \( g \) and \( h \) have continuous derivatives on \([a, b]\). Then

\[ \int_y^x g (t) \, h' (t) \, dt = [(g(t) h(t)]_y^x = \int_y^x g' (t) \, h (t) \, dt. \]

\[ (♣) \text{Example} \] Assume that \( f^{(n+1)} \) is continuous on \([a, b]\). Integrate

\[ \frac{1}{n!} \int_0^h (h - u)^n f^{(n+1)} (u) \, du \]

by parts, repeatedly.

Question ??

To use integration by parts in this example we need \( g(t) = (h - t)^n \) and \( h(t) = f^{(n)} (u) \) to both have continuous derivatives. In particular \( f^{(n+1)} \) needs to be continuous on \([0, h]\). If you look back at the various forms of the error term for Taylor’s series you will see the integral form required \( f^{(n+1)} \) to be continuous while the other two forms did not.(♣)

I leave it to the student to justify the integration of the chain rule for differentiation. This would give

Theorem Change of Variable or Integration by Substitution

Assume \( f \) and \( g' \) are continuous on \([a, b]\). Then
\[ \int_a^b f(x) \, dx = \int_{g(\alpha)}^{g(\beta)} (f \circ g)(t) \, g'(t) \, dt \]

where \( g(\beta) = b \) and \( g(\alpha) = a \).

**Proof** Let \( F \) be a primitive for \( f \). Then, by the chain rule \( F(g(x)) \) has derivative \( f(g(x))g'(x) \), i.e. it is a primitive for this. So, by the Corollary,

\[
\int_{g(\alpha)}^{g(\beta)} f(g(t)) \, g'(t) \, dt = F(g(\beta)) - F(g(\alpha)) = F(b) - F(a) = \int_a^b f(x) \, dx.
\]

**Example** With \( \ln x \) expressed as an integral, as we saw in an earlier example, prove that for all \( a, b > 0 \) we have \( \ln ab = \ln a + \ln b \).

4.5 Improper Integrals *(We may well not cover this section in lectures)*

We have only defined the integral for closed intervals and functions bounded on such intervals. We finish with a list of definitions that try to extend the situations in which our definitions are meaningful.

**Definition** If \( f : [a, \infty) \to \mathbb{R} \) is Riemann integrable on every interval \([a, b]\) and

\[
\lim_{t \to \infty} \int_a^t f(x) \, dx
\]

make sense with the limit existing, we define this limit to be \( \int_a^{\infty} f(x) \, dx \) and we say that the integral *converges*. Otherwise it *diverges*.

Similarly for \( \int_{-\infty}^b f(x) \, dx \).

**Example** Show that

\[
\int_1^{\infty} \frac{1}{x^\alpha} \, dx
\]

converges if, and only if, \( \alpha > 1 \).

(♣) This may be reminiscent of a result in course 153 concerning those \( k \) for which \( \sum_{r=1}^{\infty} \frac{1}{r^k} \) converges. But then there is often a connection between the series \( \sum_{r=1}^{\infty} f(n) \) and integral \( \int_1^{\infty} f(t) \, dt \). For instance, if \( f : [1, \infty) \to \mathbb{R} \) is a
positive decreasing function then the series and integral either both diverge or both converge. (*)

**Definition** We define \( \int_{-\infty}^{\infty} f(x) \, dx \) as

\[
\int_{-\infty}^{0} f(x) \, dx + \int_{0}^{\infty} f(x) \, dx,
\]

and it converges only if both of these do so separately.

**Definition** (Gauss, 1812) If \( f \) is defined on \((a, b]\) and \( \lim_{\alpha \to a} \int_{\alpha}^{b} f(x) \, dx \) makes sense and the limit exists, then we define the limit to be \( \int_{a}^{b} f(x) \, dx \) and say that the integral converges. Otherwise it diverges.

Similarly for a function defined on \([a, b)\).

**Example** For what \( \alpha \) does \( \int_{0}^{1} \frac{dx}{x^\alpha} \) converge? Question ??

**Definition** Cauchy Principal Value. Assume that \( f \) is not defined at \( c \) in \([a, b]\) but, for all \( \eta > 0 \), is bounded in \([a, c - \eta] \cup [c + \eta, b]\). Then the Principal Value Integral is defined to be

\[
P.V \int_{a}^{b} f(x) \, dx = \lim_{\eta \to 0} \left( \int_{a}^{c-\eta} f(x) \, dx + \int_{c+\eta}^{b} f(x) \, dx \right),
\]

provided that this limit exists.