Section 5 Series with non-negative terms

**Theorem 5.1** Let \( \sum_{r=1}^{\infty} a_r \) be a series with non-negative terms and let \( s_n \) be the \( n \)-th partial sum for each \( n \in \mathbb{N} \). Then \( \sum_{r=1}^{\infty} a_r \) is convergent if, and only if, \( \{s_n\}_{n \in \mathbb{N}} \) is bounded.

**Proof** Since \( a_r \geq 0 \) for all \( r \in \mathbb{N} \), then \( s_{n+1} - s_n = a_{n+1} \geq 0 \), i.e. \( s_{n+1} \geq s_n \) for all \( n \geq 1 \) and so the sequence \( \{s_n\}_{n \in \mathbb{N}} \) of partial sums is increasing.

\((\Rightarrow)\) If \( \sum_{r=1}^{\infty} a_r \) converges then \( \{s_n\}_{n \in \mathbb{N}} \) converges by definition. Hence, by Theorem 3.2, \( \{s_n\}_{n \in \mathbb{N}} \) is bounded.

\((\Leftarrow)\) Conversely, if \( \{s_n\}_{n \in \mathbb{N}} \) is bounded then, in particular, it is bounded above. Since \( \{s_n\}_{n \in \mathbb{N}} \) is also increasing, then \( \{s_n\}_{n \in \mathbb{N}} \) is convergent by Theorem 3.4. Thus we have verified the definition that \( \sum_{r=1}^{\infty} a_r \) is convergent. \( \blacksquare \)

**Remark** If the series of non-negative terms \( \sum_{r=1}^{\infty} a_r \) is convergent, the sequence \( \{s_n\}_{n \in \mathbb{N}} \) is convergent and its limit, which is the sum of the series, is the lub \( \{s_n : n \in \mathbb{N}\} \). (See Theorem 3.4.)

The next result is a way of testing convergence or divergence by comparison with a known series.

**Theorem 5.2** (First Comparison Test)

Let \( \sum_{r=1}^{\infty} a_r \) and \( \sum_{r=1}^{\infty} b_r \) be series with \( 0 \leq a_r \leq b_r \) for all \( r \in \mathbb{N} \).

(i) If \( \sum_{r=1}^{\infty} b_r \) is convergent then \( \sum_{r=1}^{\infty} a_r \) is convergent. If \( \sum_{r=1}^{\infty} b_r \) has sum \( \tau \) and \( \sum_{r=1}^{\infty} a_r \) has a sum \( \sigma \), then \( \sigma \leq \tau \).

(ii) If \( \sum_{r=1}^{\infty} a_r \) is divergent, then \( \sum_{r=1}^{\infty} b_r \) is divergent.

**Proof**

(i) Let \( s_n \) and \( t_n \) be the \( n \)-th partial sums of \( \sum_{r=1}^{\infty} a_r \) and \( \sum_{r=1}^{\infty} b_r \), respectively. As in the proof of Theorem 5.1 both \( \{s_n\}_{n \in \mathbb{N}} \) and \( \{t_n\}_{n \in \mathbb{N}} \) are increasing sequences.

By hypothesis, \( \{t_n\}_{n \in \mathbb{N}} \) is convergent with limit \( \tau \). But \( \{t_n\}_{n \in \mathbb{N}} \) is increasing, so by Theorem 3.4, \( \tau \) is the least upper bound of \( \{t_n : n \in \mathbb{N}\} \).

Since \( 0 \leq a_r \leq b_r \) for all \( r \in \mathbb{N} \), we have that

\[ 0 \leq \sum_{r=1}^{n} a_r \leq \sum_{r=1}^{n} b_r, \]

i.e. \( 0 \leq s_n \leq t_n \) for all \( n \in \mathbb{N} \). Thus all the \( s_n \) are no greater than any upper bound of \( \{t_n : n \in \mathbb{N}\} \), that is, \( s_n \leq \tau \) for all \( n \in \mathbb{N} \). So \( \tau \) is an upper bound
for \( \{s_n : n \in \mathbb{N}\} \).

Then, since \( \{s_n\}_{n \in \mathbb{N}} \) is also increasing, we have again by Theorem 3.4 that 
\( \{s_n\}_{n \in \mathbb{N}} \) is convergent with limit \( \sigma = \text{lub}\{s_n : n \in \mathbb{N}\} \). Being the least of all upper bounds \( \sigma \) is less than or equal to any upper bound of the \( \{s_n : n \in \mathbb{N}\} \). In particular, \( \sigma \leq \tau \).

(ii) Again, this is simply the contrapositive of part (i) (See the appendix within section 3 of these notes.) \(\blacksquare\)

Example Show that \( \sum_{r=0}^{\infty} \frac{1}{3^r+1} \) is convergent and \( \sum_{r=1}^{\infty} \frac{1}{r^{2/3}} \) is divergent.

Solution Firstly,

\[
0 \leq \frac{1}{3^r+1} \leq \frac{1}{3^r}
\]

and \( \sum_{r=0}^{\infty} \frac{1}{3^r} \) converges since it is a Geometric Series with ratio \( \frac{1}{3} \) (See Theorem 4.1). Hence our series converges.

Secondly,

\[
0 \leq \frac{1}{r} \leq \frac{1}{r^{2/3}}
\]

and the fact that \( \sum_{r=1}^{\infty} \frac{1}{r} \) diverges is an earlier example. Hence our series diverges. \(\blacksquare\)

See also Question 6 Sheet 5

**Theorem 5.3** (Second Comparison Test)

Let \( \sum_{r=1}^{\infty} a_r \) and \( \sum_{r=1}^{\infty} b_r \) be series such that \( a_r \geq 0 \) and \( b_r > 0 \) for all \( r \in \mathbb{N} \). Suppose that the sequence \( \{a_n / b_n\}_{n \in \mathbb{N}} \) is convergent with limit \( \ell \neq 0 \).

Then \( \sum_{r=1}^{\infty} a_r \) is convergent if and only if \( \sum_{r=1}^{\infty} b_r \) is convergent.

**Proof**

Suppose that \( \lim_{n \to \infty} a_n / b_n = \ell \). Since \( a_n \geq 0 \) and \( b_n > 0 \) we have \( a_n / b_n \geq 0 \) and thus \( \ell \geq 0 \). But, by assumption, \( \ell \neq 0 \), hence \( \ell > 0 \).

We now apply Lemma 3.6, concluding that there exists \( N_0 \in \mathbb{N} \) such that

\[
\frac{\ell}{2} < \frac{a_n}{b_n} < \frac{3\ell}{2}
\]  \hfill (11)

for all \( n \geq N_0 \).

(\(\Rightarrow\)) First suppose that \( \sum_{r=1}^{\infty} a_r \) is convergent.
By Theorem 4.2 $\sum_{r=N_0}^{\infty} a_r$ is convergent.

By Theorem 4.4 $\sum_{r=N_0}^{\infty} \frac{2}{r} a_r$ is convergent.

From (11) we have

$$0 < b_n < \frac{2}{\ell} a_n$$

for all $n \geq N_0$. So, by the First Comparison Test, $\sum_{r=N_0}^{\infty} b_r$ is convergent.

Finally, by Theorem 4.2 again, $\sum_{r=1}^{\infty} b_r$ is convergent.

($\Leftarrow$) Conversely, suppose that $\sum_{r=1}^{\infty} b_r$ is convergent.

By Theorem 4.2 $\sum_{r=N_0}^{\infty} b_r$ is convergent.

By Theorem 4.4 $\sum_{r=N_0}^{\infty} \frac{3\ell}{2} a_r$ is convergent.

This time we use (11) in the form

$$0 \leq a_n < \frac{3\ell}{2} b_n$$

for all $n \geq N_0$. So, by the First Comparison Test, $\sum_{r=N_0}^{\infty} \frac{3\ell}{2} a_r$ is convergent.

Again $\sum_{r=1}^{\infty} a_r$ is convergent, justified by Theorems 4.2.

\[\Box\]

**Note** If the sequence $\{a_n/b_n\}_{n \in \mathbb{N}}$ is either divergent or has a zero limit then Theorem 5.3 tells us nothing. We have to either choose a different series $\sum b_r$ for comparison or use a different test on our given series $\sum a_r$.

We can use the Comparison tests to prove the following.

**Theorem 5.4**

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is convergent.

**Solution.** As before, the idea is to compare this series with

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}.$$ 

This may not look a “simpler” series but we saw in Theorem 4.8 that it is easy to sum.

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. Then $\frac{a_n}{b_n} = 1 + \frac{1}{n}$ and so $\lim_{n \to \infty} \frac{a_n}{b_n} = 1 \neq 0$.

Hence by the Second Comparison test, Theorem 5.3, $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is convergent.
Excercise for students; try to show that $\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges, with sum less than 2, using the First Comparison Test.

Note In later courses it will be shown that $\sum_{r=1}^{\infty} \frac{1}{r^2}$ has sum $\pi^2/6$.

Theorem 5.4 For $k \in \mathbb{Z}$ we have that

$$\sum_{r=1}^{\infty} \frac{1}{r^k} \quad \text{is} \quad \begin{cases} \text{convergent if } k \geq 2 \\ \text{divergent if } k \leq 1. \end{cases}$$

Proof (Left to student)

Example Test the series $\sum_{r=1}^{\infty} \frac{2r^2 + 2r + 1}{r^5 + 2}$ for convergence.

Solution

Rough work

For large $r$, $2r^2 + 2r + 1$ is dominated by $2r^2$ (i.e. if $r = 1,000$ then $2r^2$ differs from $2r^2 + 2r + 1$ by less than 0.1%). Similarly $r^5 + 2$ is dominated by $r^5$, so for large $r$ the sum will “look like” $\sum_{r} \frac{2}{r}$ which we know, by Theorem 5.4, converges.

End of rough work

Let

$$a_n = \frac{2n^2 + 2n + 1}{n^5 + 2}, \quad \text{and} \quad b_n = \frac{1}{n^5}.$$ 

Then

$$\frac{a_n}{b_n} = \frac{n^3(2n^2 + 2n + 1)}{n^5 + 2} = \frac{2 + \frac{2}{n} + \frac{2}{n^2}}{1 + \frac{2}{n}} \quad \text{so} \quad \lim_{n \to \infty} \frac{a_n}{b_n} = 2 \neq 0.$$ 

Since, by Theorem 5.4, $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is convergent, we can use the Second Comparison Test to deduce that $\sum_{r=1}^{\infty} \frac{2r^2 + 2r + 1}{r^5 + 2}$ converges. 

Example Test the series $\sum_{r=1}^{\infty} \frac{r^2 - 2r - 3}{r^3 - 2}$ for convergence.
Proof

Rough work
For large $r$ the general term of this series will “look like” $\frac{r^2}{r^3} = \frac{1}{r}$, the sum of which we know diverges.
End of rough work

Let

$$a_n = \frac{n^2 - 2n - 3}{n^3 - 2}, \quad \text{and} \quad b_n = \frac{1}{n}. $$

Then

$$\frac{a_n}{b_n} = \frac{n(n^2 - 2n - 3)}{n^3 - 2} = \frac{1 - \frac{2}{n} - \frac{3}{n^2}}{1 - \frac{2}{n^3}},$$

so \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \neq 0 \).

Since by an example above, the Harmonic series \( \sum_{r=1}^{\infty} \frac{1}{r} \) is divergent, we can use the Second Comparison Test to deduce that \( \sum_{r=1}^{\infty} \frac{r^2 - 2r - 3}{r^3 - 2} \) diverges. \( \blacksquare \)

Exercise for student: try to prove the last result using the First Comparison Test.

Remark In the last example we have cheated slightly as \( a_r < 0 \) when \( r = 2 \). The Comparison Test requires \( a_r \geq 0 \) for all \( r \). However, this does not matter because we can apply the test to \( \sum_{r=3}^{\infty} a_r \) and deduce that this is divergent. Then \( \sum_{r=1}^{\infty} a_r \) must also be divergent. Thus the Comparison Tests can be applied to series \( \sum_{r=1}^{\infty} a_r \) which have at most a finite number of negative terms.

Appendix

Theorem 5.5 For \( k \in \mathbb{Z} \) we have that

$$\sum_{r=1}^{\infty} \frac{1}{r^k} \quad \text{is} \quad \begin{cases} \text{convergent if } k \geq 2 \\ \text{divergent if } k \leq 1. \end{cases}$$

Proof If \( k \geq 2 \) then

$$0 < \frac{1}{r^k} \leq \frac{1}{r^2}.$$

for all \( r \in \mathbb{N} \). By Theorem 5.4, \( \sum_{r=1}^{\infty} \frac{1}{r^2} \) is convergent. So by the First Comparison Test, Theorem 5.2, we deduce that \( \sum_{r=1}^{\infty} \frac{1}{r^k} \) is convergent.

If \( k \leq 1 \) then
\[ \frac{1}{r} \leq \frac{1}{r^k} \]

for all \( r \in \mathbb{N} \). We have seen earlier that the Harmonic series, \( \sum_{r=1}^{\infty} \frac{1}{r} \), is divergent. So by the First Comparison Test, Theorem 5.2, we deduce that \( \sum_{r=1}^{\infty} \frac{1}{r^k} \) is divergent. ■

Note I have restricted to \( k \in \mathbb{Z} \) in Theorem 5.5 since I have not defined \( r^k \) when \( r \in \mathbb{N} \), for a general \( k \in \mathbb{R} \). For example, how would we define \( 2^{\sqrt{2}} \) or \( 3^{\pi} \)?

But we can define \( r^k \) when \( k \in \mathbb{Q} \). For when \( k \in \mathbb{Q} \) we can write \( k = p/q \) where \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \). Then we can define \( r^k = (r^{1/q})^p \) where \( r^{1/q} \) is the positive real root of \( x^q - r = 0 \).

With this definition we can extend Theorem 5.5: Let \( k \in \mathbb{Q} \). Then

\[
\sum_{r=1}^{\infty} \frac{1}{r^k}
\]

is \( \begin{cases} 
\text{convergent if } k > 1 \\
\text{divergent if } k \leq 1.
\end{cases} \)

This shows that the case \( k = 1 \), the Harmonic series, is on the boundary between convergence and divergence. In particular, it diverges but it does so slowly.