Section 3 Sequences and Limits

**Definition** A **sequence** of real numbers is an infinite ordered list \(a_1, a_2, a_3, a_4, \ldots\) where, for each \(n \in \mathbb{N}\), \(a_n\) is a real number. We call \(a_n\) the \(n\)-th term of the sequence.

Usually (but not always) the sequences that arise in practice have a recognisable pattern and can be described by a formula.

**Examples** Find a formula for \(a_n\) in each of the following cases:

(i) \(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\) \(a_n = \frac{1}{n}\) for all \(n \in \mathbb{N}\),

(ii) \(1, -1, 1, -1, \ldots\) \(a_n = (-1)^n+1\) for all \(n \in \mathbb{N}\),

(iii) \(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\) \(a_n = \frac{2^n - 1}{2^n}\) for all \(n \in \mathbb{N}\),

(iv) \(2, 2, 0, 0, 0, 0, 0, 0, \ldots\) \(a_n = 2\) if \(n \leq 3\), \(a_n = 0\) if \(n \geq 4\),

(v) \(1, 1, 2, 3, 5, 8, 13, \ldots\) \(a_n = a_{n-1} + a_{n-2}\) for all \(n \geq 3\), along with \(a_1 = a_2 = 1\).

See also Question 5 Sheet 2.

Conversely we can define a sequence by a formula.

**Example** Let

\[
a_n = \begin{cases} 
2^n & \text{if } n \text{ odd} \\
\frac{n}{n+1} & \text{if } n \text{ even}
\end{cases}
\]

for all \(n \in \mathbb{N}\).

Then we get the sequence 2, 2, 8, 4, 32, 6, \ldots.

**Exercise** for student: Show this formula can be written as

\[
a_n = \left(\frac{1 + (-1)^n}{2}\right) n + \left(\frac{1 + (-1)^n}{2}\right) 2^n.
\]

**Note** A sequence is different to a set of real numbers - the order of the terms is important in a sequence but irrelevant in a set. For instance, the sequence \(1, \frac{1}{3}, \frac{1}{9}, \frac{1}{5}, \ldots\) is different from the sequence \(\frac{1}{3}, 1, \frac{1}{3}, \frac{1}{5}, \ldots\), even though the sets \(\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{5}, \ldots\}\) and \(\{\frac{1}{3}, 1, \frac{1}{3}, \frac{1}{5}, \ldots\}\) are identical.

We denote a sequence \(a_1, a_2, a_3, \ldots\) by \(\{a_n\}_{n \in \mathbb{N}}\) or \(\{a_n\}_{n \geq 1}\) or just \(\{a_n\}\) if there is no confusion. For example \(\{\frac{2^n - 1}{2^n}\}\) is sequence (iii) above.

The set containing the sequence is written as \(\{a_n : n \in \mathbb{N}\}\).
**Definition** A real number $\ell$ is said to be a *limit* of a sequence $\{a_n\}_{n \in \mathbb{N}}$ if, and only if,

for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - \ell| < \varepsilon$ for all $n \geq N$

or, in mathematical notation,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - \ell| < \varepsilon.$$  

**Note.** To check, or verify that this definition holds we have to:

(i) Guess the value of the limit $\ell$,

(ii) Assume $\varepsilon > 0$ has been given,

(iii) Find $N \in \mathbb{N}$ such that $|a_n - \ell| < \varepsilon$, i.e. $\ell - \varepsilon < a_n < \ell + \varepsilon$ for all $n \geq N$.

We have to be able to find such an $N$ for each and every $\varepsilon > 0$ and, in general, the $N$ will depend on $\varepsilon$. So you will often see $N$ written as a function of $\varepsilon$, i.e. $N(\varepsilon)$. 

See Questions 8 and 9 Sheet 2

**Definition** A sequence which has a limit is said to be *convergent*. A sequence with no limit is called *divergent*.

**Example** The sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ is convergent with limit 0.

**Solution** This is simply the Archimedean Principle. We have to verify the definition above with $\ell = 0$.

Let $\varepsilon > 0$ be given. (So we have no choice over $\varepsilon$, it can be any such number.)

The Archimedean Principle says that we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. But then, for all $n \geq N$ we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence we have verified the definition with $\ell = 0$ which must, therefore, be a limit of the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$. ■

The question remains whether the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ has other limits. Note how in the definition I talked about $\ell$ being a limit, not the limit. The following result answers this in the negative.
Theorem 3.1 If a sequence of real numbers \( \{a_n\}_{n \in \mathbb{N}} \) has a limit, then this limit is unique.

**Proof** by contradiction. We hope to prove “For all convergent sequences the limit is unique”.

The negation of this is “There exists at least one convergent sequence which does not have a unique limit”.

This is what we assume.

On the basis of this assumption let \( \{a_n\}_{n \in \mathbb{N}} \) denote a sequence with more than one limit, two of which are labelled as \( \ell_1 \) and \( \ell_2 \) with \( \ell_1 \neq \ell_2 \).

Choose \( \varepsilon = \frac{1}{3} |\ell_1 - \ell_2| \) which is greater than zero since \( \ell_1 \neq \ell_2 \).

Since \( \ell_1 \) is a limit of \( \{a_n\}_{n \in \mathbb{N}} \) we can apply the definition of limit with our choice of \( \varepsilon \) to find \( N_1 \in \mathbb{N} \) such that

\[
|a_n - \ell_1| < \varepsilon \quad \text{for all } n \geq N_1.
\]

Similarly, as \( \ell_2 \) is a limit of \( \{a_n\}_{n \in \mathbb{N}} \) we can apply the definition of limit with our choice of \( \varepsilon \) to find \( N_2 \in \mathbb{N} \) such that

\[
|a_n - \ell_2| < \varepsilon \quad \text{for all } n \geq N_2.
\]

(There is no reason to assume that in the two uses of the definition of limit we should find the same \( N \in \mathbb{N} \) for the different \( \ell_1 \) and \( \ell_2 \). They may well be different which is why I have labelled them differently as \( N_1 \) and \( N_2 \).)

Choose any \( m_0 > \max(N_1, N_2) \), then \( |a_{m_0} - \ell_1| < \varepsilon \) and \( |a_{m_0} - \ell_2| < \varepsilon \). This shows that \( \ell_1 \) is “close to” \( a_{m_0} \) and \( \ell_2 \) is also “close to” \( a_{m_0} \). Hence we must have that \( \ell_1 \) is “close to” \( \ell_2 \). Using the Triangle inequality, Theorem 1.2, we can remove the \( a_{m_0} \) in the following way: (TRICK)

\[
|\ell_1 - \ell_2| = |\ell_1 - a_{m_0} + a_{m_0} - \ell_2| = |\ell_1 - a_{m_0}| + |a_{m_0} - \ell_2| < \varepsilon + \varepsilon \\
< 2\varepsilon, \quad \text{by the choice of } m_0, \\
= 2\varepsilon = \frac{2}{3}|\ell_1 - \ell_2|, \quad \text{by the definition of } \varepsilon.
\]

So we find that \( |\ell_1 - \ell_2| \), which is not zero, satisfies \( |\ell_1 - \ell_2| < \frac{2}{3}|\ell_1 - \ell_2| \), which is a contradiction.

Hence our assumption must be false, that is, there does not exist a sequence with more than one limit. Hence for all convergent sequences the limit is unique. \( \blacksquare \)
Notation Suppose \( \{a_n\}_{n \in \mathbb{N}} \) is convergent. Then by Theorem 3.1 the limit is unique and so we can write it as \( \ell \), say. Then we write \( \lim_{n \to \infty} a_n = \ell \) or \( L_{n \to \infty} a_n = \ell \) or \( a_n \to \ell \) as \( n \to \infty \).

In particular, the above example shows that

\[
\lim_{n \to \infty} \frac{1}{n} = 0.
\]

Example What is the limit of \( \{1 + \left(-\frac{1}{2}\right)^n\}_{n \in \mathbb{N}} \)?

Solution Rough work

The first few terms are: \( \frac{1}{2}, \frac{7}{8}, \frac{17}{16}, \frac{31}{32}, \ldots \).

It appears that the terms are getting closer to 1.

To prove this we have to consider

\[
|a_n - 1| = \left| \left(1 + \left(-\frac{1}{2}\right)^n\right) - 1 \right| = \left(\frac{1}{2}\right)^n.
\]

Let \( \varepsilon > 0 \) be given. We have to show that there exists \( N \in \mathbb{N} \) such that \( |a_n - 1| < \varepsilon \) for all \( n \geq N \).

Consider some particular choices of \( \varepsilon \).

\[
\begin{align*}
\varepsilon &= \frac{1}{10} : & \text{for all } n \geq 4 & |a_n - 1| = \frac{1}{2^n} \leq \frac{1}{10} < \varepsilon, \\
\varepsilon &= \frac{1}{100} : & \text{for all } n \geq 7 & |a_n - 1| < \frac{1}{128} < \varepsilon, \\
\varepsilon &= \frac{1}{1000} : & \text{for all } n \geq 10 & |a_n - 1| < \frac{1}{1024} < \varepsilon.
\end{align*}
\]

Note how these values of \( N \), namely 4, 7, 10, etc., get larger as \( \varepsilon \) gets smaller.

End of rough work

Completion of solution. By the Archimedean property we can find \( N \in \mathbb{N} \) such that \( \frac{1}{N} < \varepsilon \). For any \( n \in \mathbb{N} \) we have \( 2^n > n \) and so, for all \( n \geq N \) we have

\[
|a_n - 1| = \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon
\]

as required.
Examples Discuss the convergence or otherwise of the following sequences.

(i) \(2, 2, 2, \ldots\), convergent limit 2,
(ii) \(2\frac{1}{2}, 2\frac{1}{7}, 2\frac{1}{4}, \ldots\), convergent limit 2,
(iii) \(3 + 2, 3 - \frac{3}{2}, 3 + \frac{2}{3}, 3 - \frac{2}{4}, \ldots\), convergent limit 3,
(iv) \(1, 2, 1, 2, \ldots\), divergent,
(v) \(\frac{1}{2}, 1\frac{1}{2}, \frac{1}{3}, 1\frac{1}{3}, \frac{1}{4}, 1\frac{1}{4}, \ldots\), divergent,
(vi) \(2, 4, 6, 8, \ldots\), divergent,
(vii) \(-1, -4, -9, -25, \ldots\) divergent.

Example Show, by using the Archimedean principle to verify the definition, that sequence (iii) has limit 3.

Solution
Rough work
The \(n^{th}\) term can be written as
\[ a_n = 3 + \frac{(-1)^{n+1}2}{n} \]
So, \(|a_n - 3| = \frac{2}{n}\). We will want to find \(N \in \mathbb{N}\) such that \(\frac{2}{n} < \varepsilon\) for all \(n \geq N\), i.e. \(\frac{1}{n} < \frac{\varepsilon}{2}\) for such \(n\). Again we will do this by the Archimedean Principle.

End of Rough work

Proof
Let \(\varepsilon > 0\) be given. By the Archimedean property we can find \(N \in \mathbb{N}\) such that \(\frac{1}{N} < \frac{\varepsilon}{2}\). Then for all \(n \geq N\) we have
\[ |a_n - 3| = \frac{2}{n} \leq \frac{2}{N} < \varepsilon \]
as required.

Definition A sequence \(\{a_n\}_{n \in \mathbb{N}}\) is said to be bounded if the set \(\{a_n : n \in \mathbb{N}\}\) = \(\{a_1, a_2, a_3, a_4, \ldots\}\) is bounded.

Similarly a sequence is said to be bounded above or bounded below if the set is bounded above or bounded below respectively.

Example 1, 2, 1, 2, 1, 2... is a bounded sequence.
Theorem 3.2 If \( \{a_n\}_{n \in \mathbb{N}} \) is a convergent sequence, then \( \{a_n\}_{n \in \mathbb{N}} \) is a bounded sequence.

Proof
Let \( \ell \) be the limit of \( \{a_n\}_{n \in \mathbb{N}} \). In the definition of limit choose \( \varepsilon = 1 \) to find \( N \in \mathbb{N} \) such that \( |a_n - \ell| < 1 \) for all \( n \geq N \). Rewriting, this says that
\[
\ell - 1 < a_n < \ell + 1, \quad \text{for all } n \geq N,
\]
or that the set \( \{a_N, a_{N+1}, a_{N+2}, \ldots\} \) is bounded.

Yet the set \( \{a_1, a_2, a_3, \ldots a_{N-1}\} \) is bounded, above by \( \max \{a_i : 1 \leq i \leq N - 1\} \) and from below by \( \min \{a_i : 1 \leq i \leq N - 1\} \). These maximum and minimums can be calculated simply because the set is finite.

If \( A, B \) are bounded sets then \( A \cup B \) is bounded.
(Exercise, prove this, but see also Question 3, sheet 2)

Hence \( \{a_1, a_2, a_3, \ldots a_{N-1}\} \cup \{a_N, a_{N+1}, a_{N+2}, \ldots\} = \{a_1, a_2, a_3, \ldots\} \) is bounded as is, therefore, the original sequence. \( \blacksquare \)

Corollary 3.3 If \( \{a_n\}_{n \in \mathbb{N}} \) is an unbounded sequence, then \( \{a_n\}_{n \in \mathbb{N}} \) is divergent.

Proof: This is just a restatement of Theorem 3.2.

The statement of Theorem 3.2 is of the form “If \( p \) then \( q \)”, often written as “\( p \Rightarrow q \)”. This has been discussed in the appendix to part 2. We also saw there that we represent the negation of a proposition \( p \) as \( \neg p \). In other words, \( \neg p \) means that \( p \) does not hold.

If we had both \( p \Rightarrow q \) and \( \neg q \Rightarrow p \) we could combine to deduce \( \neg q \Rightarrow p \Rightarrow q \), i.e. \( \neg q \Rightarrow q \). It would be a strange world if, assuming that \( q \) does not hold we could then deduce that \( q \) did hold. For this reason we say that \( p \Rightarrow q \) and \( \neg q \Rightarrow p \) are inconsistent.

Without proof I state that \( p \Rightarrow q \) and \( \neg q \Rightarrow \neg p \) are consistent. In fact they are logically equivalent in that if one statement is false than so is the other and if one is true then so is the other. See the appendix to part 2 for more details of equivalence. We say that \( \neg q \Rightarrow \neg p \) is the contrapositive of \( p \Rightarrow q \). The statement of Corollary 3.3 is simply the contrapositive of Theorem 3.2. \( \blacksquare \)

Example The sequence \( 1 \frac{1}{2}, 2 \frac{1}{3}, 3 \frac{1}{4}, 4 \frac{1}{5}, \ldots \) is not bounded above and thus it is divergent.

Proof by contradiction.
Assume the sequence is bounded above by \( \lambda \), say. By the alternative Archimedean principle, Theorem 2.1', we can find \( n \in \mathbb{N} \) such that \( n > \lambda \).
But then \( n + \frac{1}{n+1} \) is an element of the sequence satisfying \( n + \frac{1}{n+1} > n > \lambda \), which is a contradiction.

Hence our assumption is false, thus the sequence is not bounded above.\( \blacksquare \)

**Definition** A sequence \( \{b_n\}_{n \in \mathbb{N}} \) is called a subsequence of \( \{a_n\}_{n \in \mathbb{N}} \) if, and only if, all of the terms of \( \{b_n\}_{n \in \mathbb{N}} \) occur amongst the terms of \( \{a_n\}_{n \in \mathbb{N}} \) in the same order.

**Examples**

(i) \( a_n = \frac{1}{n} \), \( b_n = \frac{1}{2n} \), so

\[
\{a_n\}_{n \in \mathbb{N}} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \right\}
\]

and

\[
\{b_n\}_{n \in \mathbb{N}} = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots \right\}
\]

which is a subsequence of \( \{a_n\}_{n \in \mathbb{N}} \).

(ii) \( \frac{31}{32}, \frac{63}{64}, \frac{127}{128}, \ldots \), is a subsequence of \( \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128}, \ldots \).

(iii) \( \frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots \), is not a subsequence of \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \).

**Notes** (a) We can look upon a subsequence \( \{b_n\}_{n \in \mathbb{N}} \) as the original sequence, \( \{a_m\}_{m \in \mathbb{N}} \), with terms deleted and the remaining ones relabelled. For example:

\[
\begin{array}{cccccccc}
  a_1 & a_2 & a_3 & a_4 & a_5 & \ldots & a_{m-1} & a_m & a_{m+1} & \ldots \\
  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
  b_1 & b_2 & b_3 & \ldots & b_n & b_{n+1} & \ldots
\end{array}
\]

From this we can see that each \( b_n \) comes from some \( a_m \) where \( n \) and \( m \) satisfy

\[
m = n + (\text{the number of } a_i, 1 \leq i \leq m - 1, \text{that have been deleted}).
\]

In particular \( m \geq n \). Hence we have

\[
\forall n \geq 1, \exists m \geq n : b_n = a_m.
\]
The fact that the relabelling retains the ordering means that if \( b_n = a_m \) and \( b_{n'} = a_{m'} \) then \( n \geq n' \) if, and only if, \( m \geq m' \).

(b) Example (ii) illustrates the common method of forming a subsequence by omitting a finite number of initial terms of a given sequence.

**Theorem 3.4** If a sequence converges then all subsequences converge and all convergent subsequences converge to the same limit.

**Proof** Let \( \{a_n\}_{n \in \mathbb{N}} \) be any convergent sequence. Denote the limit by \( \ell \).

Let \( \{b_n\}_{n \in \mathbb{N}} \) be any subsequence.

Let \( \varepsilon > 0 \) be given. By the definition of convergence for \( \{a_n\}_{n \in \mathbb{N}} \) there exists \( N \in \mathbb{N} \) such that \( |a_n - \ell| < \varepsilon \) for all \( n \geq N \). But this value \( N \) will also work for \( \{b_n\}_{n \in \mathbb{N}} \). This is because if \( n \geq N \) then \( b_n = a_m \) for some \( m \geq n \geq N \) and so \( |b_n - \ell| = |a_m - \ell| < \varepsilon \). Thus \( |b_n - \ell| < \varepsilon \) for all \( n \geq N \) as required.

**Question** What is the contrapositive of Theorem 3.4?

**Question** What is the negation of “all subsequences converge and all convergent subsequences converge to the same limit.”?

In logic, if it is not the case that both \( p \) and \( q \) holds then either \( p \) does not hold or \( q \) does not hold. We could write this as saying “not \( (p \text{ and } q) \)” is logically equivalent to “either (not \( p \)) or (not \( q \))”. Thus, the negation of “all subsequences converge and all convergent subsequences converge to the same limit” is “either (not all subsequences converge) or (not all convergent subsequences have the same limit)”. This is the same as “either (there exists a diverging subsequence) or (there are two converging subsequences with different limits).”

So the contrapositive of Theorem 3.4 is:

**Corollary 3.5** If \( \{a_n\}_{n \in \mathbb{N}} \) is a sequence that either has a subsequence that diverges or two convergent subsequences with different limits then \( \{a_n\}_{n \in \mathbb{N}} \) is divergent.

**Example** The sequence 1, 2, 1, 2, 1, 2, ... is divergent.

**Solution** Consider the two subsequences 1, 1, 1, ... and 2, 2, 2, ..., both convergent though with different limits, 1 and 2. Hence by the Corollary the sequence 1, 2, 1, 2, 1, 2, ... diverges.

**Example** The sequence 1, 2, 3, 1, 2, 3, 1, 2, 3, ... is divergent.

**Solution** Our sequence has a subsequence 1, 2, 1, 2, 1, 2, ... which, by the previous example, is divergent. Hence by the Corollary the sequence 1, 2, 3, 1, 2, 3, 1, 2, 3, ... diverges.
Note The sequence 1, 2, 3, 1, 2, 3, 1, 2, 3, ... is bounded but divergent. Thus, \{a_n \}_{n \in \mathbb{N}} being bounded doesn’t necessarily mean it is convergent.

Remember these results as

| sequence convergent ⇒ sequence bounded, but |
| sequence bounded ⇒ sequence convergent. |

Aside Something you might try to prove

Theorem Every bounded sequence has a convergent subsequence.
Proof Not given

End of aside.

Definition A sequence \{a_n \}_{n \in \mathbb{N}} is said to be increasing (or non-decreasing) if \( a_n \leq a_{n+1} \) for all \( n \in \mathbb{N} \). (So \( a_1 \leq a_2 \leq a_3 \leq a_4 \leq ... \).)

A sequence \{a_n \}_{n \in \mathbb{N}} is said to be decreasing (or non-increasing) if \( a_n \geq a_{n+1} \) for all \( n \in \mathbb{N} \). (So \( a_1 \geq a_2 \geq a_3 \geq a_4 \geq ... \).)

A monotone sequence is one that is either increasing or decreasing.

A sequence is strictly increasing if \( a_n < a_{n+1} \) for all \( n \in \mathbb{N} \), is strictly decreasing if \( a_n > a_{n+1} \) for all \( n \in \mathbb{N} \) and is strictly monotone if it is either strictly increasing or strictly decreasing.

Theorem 3.6 Let \{a_n \}_{n \in \mathbb{N}} be a increasing sequence which is bounded above. Then the sequence converges with limit lub\{a_n : n \in \mathbb{N}\}.

Proof The set \{a_n : n \in \mathbb{N}\} is non-empty is bounded above by the assumption of the theorem. So, by the Completeness of \( \mathbb{R} \), Property 10, the set has a least upper bound. Denote lub\{a_n : n \in \mathbb{N}\} by \( \beta \).

We have to verify the definition of convergence with limit \( \beta \).

Let \( \varepsilon > 0 \) be given. By Theorem 2.2 there exists \( N \in \mathbb{N} \) such that \( \beta - \varepsilon < a_N \).

(In words: \( \beta \) is the least of all upper bounds, but \( \beta - \varepsilon \) is less than \( \beta \) so cannot be an upper bound and thus must be less than some element in the set.)

Since the sequence is increasing we have

\[ \beta - \varepsilon < a_N < a_{N+1} < a_{N+2} < ... \]
that is, \( \beta - \varepsilon < a_n \) for all \( n \geq N \).

But \( \beta \) is an upper bound for the set so

\[
\beta - \varepsilon < a_n \leq \beta < \beta + \varepsilon \quad \text{or} \quad |a_n - \beta| < \varepsilon
\]

for all \( n \geq N \).

Thus we have verified the definition of convergence with limit \( \beta = \operatorname{lub}\{a_n : n \in \mathbb{N}\} \).

**Theorem 3.7** Let \( \{a_n\}_{n \in \mathbb{N}} \) be a decreasing sequence which is bounded below. Then \( \operatorname{glb}\{a_n : n \in \mathbb{N}\} \) is the limit of \( \{a_n\} \) and so, in particular, \( \{a_n\}_{n \in \mathbb{N}} \) is convergent.

**Proof**

Similar to that of Theorem 3.6 and is left as an exercise.

**Example** Let \( a_n = \frac{n}{n+1} \) for all \( n \). Show that \( \{a_n\}_{n \in \mathbb{N}} \) is convergent.

**Solution**

Rough work.

Looking at the first few terms \( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \) they appear to be getting larger.

So we might hope to prove

\[
\frac{n}{n+1} \leq \frac{n+1}{n+2},
\]

i.e. \( n(n+2) \leq (n+1)^2 \) or \( n^2 + 2n \leq n^2 + 2n + 1 \) which is obviously true. We then have to show that the sequence is bounded above and we might guess by 1. So we need \( \frac{n}{n+1} \leq 1 \), i.e. \( n \leq n + 1 \), again true.

(Again this is not a proof since we have started with what we wanted to prove, deducing true statements, which is the wrong way round.)

End of rough work.

**Proof**

For all \( n \in \mathbb{N} \) we have

\[
0 < 1 \\
\Rightarrow n^2 + 2n \leq n^2 + 2n + 1 \\
\Rightarrow n(n+2) \leq (n+1)^2 \\
\Rightarrow \frac{n}{n+1} \leq \frac{n+1}{n+2}
\]

Hence the sequence is increasing.

Also, for all \( n \in \mathbb{N} \) we have \( n \leq n + 1 \) in which case \( \frac{n}{n+1} \leq 1 \). Hence the
sequence is bounded above.

Therefore, by Theorem 3.6 the sequence is convergent. ■

**Note** Using this method we have not found the value of the limit. To do so, we would have to calculate \( \text{lub} \{n/(n+1) : n \in \mathbb{N}\} \). The strength of using either Theorem 3.6 or 3.7 is that we do not need to guess the value of the limit.
Appendix

In the appendix to part 2 we discussed “if $p$ then $q$” or “$p \Rightarrow q$” when $p$ and $q$ are propositions. I said there that the compound proposition $p \Rightarrow q$ is false only when $p$ is True and $q$ False (we never want something false to follow from something true). In all other cases $p \Rightarrow q$ is defined to be True.

Consider now the contrapositive, “if not $q$ then not $p$”, or “$(\neg q) \Rightarrow (\neg p)$”. When is this False? It is False iff $\neg q$ is True and $\neg p$ False, i.e. iff $q$ is False and $p$ True, i.e. iff $p \Rightarrow q$ is False. So $p \Rightarrow q$ and $(\neg q) \Rightarrow (\neg p)$ are equivalent in that whatever truth values are given to $p$ and $q$ these two compound propositions have the same truth value.

Note, the converse of “if $p$ then $q$” is “if $q$ then $p$”, i.e. the converse of $p \Rightarrow q$ is $q \Rightarrow p$. These are not equivalent. For instance, if $p$ is True and $q$ is False then $p \Rightarrow q$ is False while $q \Rightarrow p$ is True.