

Computing Interior Eigenvalues of Conservative Gyroscopic Problems

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We propose a new method for computing a large number of eigenpairs in the interior of the spectrum for conservative gyroscopic eigenproblems by the nonlinear Arnoldi method combined with a localized version of safeguarded iteration.

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1 Introduction

In this paper we consider the conservative gyroscopic eigenvalue problem

$$T(\omega)x := \omega^2 Mx - i\omega Gx - Kx = 0 \quad (1)$$

where K is the stiffness matrix modified by the presence of centrifugal forces, M is the mass matrix, and G is the gyroscopic matrix. K and M are assumed to be symmetric and positive definite, and G is skew-symmetric.

This problem arises for instance from simulation of sound radiation of rolling tires which is the major source of traffic noise at speed above 40 km/h for passenger cars, while for trucks this limit is approximately 60 km/h (cf. [2]). Simulating the structural dynamics of the rolling tire by an Arbitrary Lagrangian Eulerian (ALE) approach developed by Nackenhorst [2] one ends up with problem (1), where a large number of eigenfrequencies in the range from 500 Hz to 2000 Hz have to be computed since this is an interval of particular perception of the human ear.

Arnoldi's method [3], [4] with safeguarded iteration for solving the projected eigenproblems is a reliable method for determining all eigengenpairs of (1) at the lower end of the (positive) spectrum. However, this approach has the disadvantage, that the dimension of the search space has to be at least m when computing the m smallest eigenvalue which is prohibitive if a large number of eigenvalues or eigenvalues in the interior of the spectrum are wanted. In this paper we propose a method to determine eigenvalues in the interior of the spectrum by Arnoldi's method in a systematic way requiring much smaller search spaces.

2 Arnoldi method for gyroscopic problems

Arnoldi's method for the nonlinear eigenproblem $T(\omega)x = 0$ is an iterative projection where the search space is expanded by the direction suggested by residual inverse iteration. For linear eigenproblems it reduces to the shift-and-invert Arnoldi method which explains its name for the general case although no Krylov space and no Arnoldi recurrence occur.

For computing all eigenvalues of the gyroscopic problem between the m_{\min} -th and m_{\max} -th smallest eigenvalue it obtains the following form:

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1: Start with an initial shift  $\sigma$  and initial orthonormal basis  $V$ 
2: determine preconditioner  $M \approx T(\sigma)^{-1}$ 
3: for  $m = m_{\min}, \dots, m_{\max}$  do
4:   compute  $m$ -th smallest eigenvalue  $\mu$  of projected problem  $V^H T(\omega) V y = 0$ 
5:   determine Ritz vector  $u = Vy$  and residual  $r = T(\mu)u$ 
6:   if  $\|r\|/\|u\| < \varepsilon$  then
7:     accept eigenpair  $\omega_m = \mu, x_m = u$ ; set  $m = m + 1$ 
8:     choose new shift  $\sigma$  and update preconditioner  $M$  if indicated
9:     restart if necessary
10:  end if
11:   $v = Mr; v = v - VV^T v; v = v/\|v\|; V = [V, v]$ ; reorthogonalize if necessary
12: end while

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In [3], [4] we commented on most of the steps of the nonlinear Arnoldi algorithm. Here we concentrate on the solution of the projected eigenproblem in step 4.

The positive eigenvalues of (1) satisfy a minmax characterization [1], and ω_m is the m smallest of them if and only if $\mu = 0$ is the m largest eigenvalue of the linear problem $T(\omega_m)y = \mu y$. This suggests the following safeguarded iteration for computing the m smallest positive eigenvalue and corresponding eigenvector, which converges locally and quadratically for simple eigenvalues. For the smallest eigenvalue the convergence is even global.

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for $k = 1, 2, \dots$ until convergence **do**

determine an eigenvector y_k corresponding to an m -largest eigenvalue of $T(\mu_k)y = \kappa y$

Compute μ_{k+1} such that $y_k^H T(\mu_{k+1})y_k = 0$

end for

The projected eigenproblem $V^H T(\omega)V y = 0$ in step 4 of the Arnoldi method inherits the symmetry from problem (1), and therefore its eigenpairs can be determined by safeguarded iteration as well. If the search space V contains eigenvectors x_1, \dots, x_m of problem (1) corresponding to the m smallest eigenvalues then the enumerations of the eigenvalues of (1) and of the projected problem $V^H T(\omega)V y = 0$ coincide up to the m smallest. Hence, the eigenvalues of problem (1) can be determined (quite) safely one after the other by Arnoldi's method combined with safeguarded iteration. A disadvantage, however, is that the search space V must be expanded at least up to dimension m if we want to determine the m -th eigenvalue this way prohibiting this approach if one has to compute a large number of eigenvalues in the interior of the spectrum.

A way out is to localize the enumeration of the eigenvalues in the following sense. Assume that we are given an eigenpair $(\hat{\omega}, \hat{x})$ of (1). Let V be a search space such that $\hat{x} \in V$. Then $\hat{\omega}$ is an eigenvalue of the projected problem $V^H T(\hat{\omega})V y = 0$ as well, and therefore $\mu = 0$ is an eigenvalue of the linear problem $V^H T(\hat{\omega})V y = \mu y$. If $\mu = 0$ is the m largest eigenvalue then we assign in this step m to $\hat{\omega}$ as its number, and we expand the search space V by the direction suggested by residual inverse iteration for computing the $(m + 1)$ -th eigenvalue. V is expanded further in the same way until convergence to the eigenpair following $\hat{\omega}$, and thereafter the Arnoldi process continues computing further eigenpairs with the same anchor $(\hat{\omega}, \hat{x})$ until the dimension of the search space exceeds a predefined bound. Then the Arnoldi method is restarted choosing the last computed eigenpair as new anchor until all wanted eigenvalues have been computed.

Notice, that the projected problem may have spurious eigenvalues which do not approximate eigenvalues of the original problem but are generated by linear combinations of eigenvectors of (1) corresponding to large and to small eigenvalues contained in V . Spurious eigenvalues will not be accepted by the Arnoldi method as approximations to eigenpairs of (1) since the residual of the corresponding Ritz pair will not be small.

Spurious eigenvalues perturb the numeration of the eigenvalues of the projected problem, and therefore, the method may return an eigenvalue approximation which has been found already in a previous step with a lower (local) number. However, due to the minmax principle a spurious eigenvalue decreases as the Arnoldi method continues and the search space grows, and eventually it will become smaller than the anchor $\hat{\omega}$. Hence, if the method detects an eigenvalue twice, and this is not a multiple eigenvalue, it is just dropped and the Arnoldi process is continued. If this eigenvalue is returned too often we restart the Arnoldi method with a new anchor corresponding to the largest accepted eigenvalue approximation.

As a spurious eigenvalue crosses the anchor the number of the anchor as an eigenvalue of the projected problem increases by one. Hence, the number of an anchor is not fixed during the iteration but has to be determined in every step of the Arnoldi method.

3 Numerical Example

To test the method we determined all eigenpairs (ω, x) of a coarse finite element model of a rotating wheel of dimension 1728 such $11800 \leq \omega \leq 16825$. These are the 101-st to 200-th eigenvalues of our problem. The experiments were run under MATLAB 6.5 on an Intel Centrino M processor with 1.7 GHz and 1 GB RAM.

Computing all eigenpairs for positive eigenvalues less than 16825 by the Arnoldi method with safeguarded iteration without restart a search space of dimension 877 is generated. The total CPU time is 11417 seconds, 11015 seconds of which are consumed to solve the projected nonlinear eigenproblems. Restarting the Arnoldi method with an orthonormal basis of the space spanned by already obtained eigenvectors if the dimension of the search space exceeds 210 the method requires 468.1 seconds to find all eigenvalues, where 231.6 seconds are needed to solve the projected eigenproblems by safeguarded iteration. We modified our strategy and restarted in the same way as before if the dimension of the search space exceeded the number of the currently converged eigenvalue by more than 30 which reduced the total CPU time further to 392.4 seconds including 150.1 seconds for the solver of the projected problems. Using the local enumeration of the eigenvalues proposed in this paper the Arnoldi method found all 100 eigenvalues in the given interval requiring 183.9 seconds, where only 10.7 seconds were needed to solve the projected eigenvalue problems.

References

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