On the McKean game driven by a spectrally negative Lévy process

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(Joint work in progress with Erik Baurdoux, LSE, UK)

Framework of a Dynkin game

- Filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $\mathbf{F}$ satisfying the usual conditions. Let τ and σ denote $\mathbf{F}$-stopping times
- Two processes L ('lower payoff') and U ('upper payoff') with $L_t \leq U_t$, $\forall t \in [0, T)$ and $L_T = U_T$
- A *maximizer* choosing σ (max. expected lower payoff), a *minimizer* choosing τ (min. expected upper payoff). Resulting payoff to the maximizer is:

$$R_{\sigma,\tau} := \mathbf{1}_{\{\sigma\leq\tau\}} L_\sigma + \mathbf{1}_{\{\tau<\sigma\}} U_\tau$$

(i.e. whoever exercises first determines the payoff)

Specifics McKean game driven by spectr. neg. Lévy process

The McKean game:

- Let $S = \exp(X)$ for a *spectr. neg. LP* X and let $\mathbb{P}_s$ be such that $\mathbb{P}_s(S_0 = s) = 1$
- Lower payoff given by $L_t = e^{-rt}(K - S_t)^+$
- Upper payoff given by $U_t = e^{-rt}\left((K - S_t)^+ + \delta\right), t \in [0, T)$ and $L_T = U_T$, where $K, r, \delta > 0$
- Assume $0 \leq \psi(1) \leq r$ (where ψ Laplacian of X)
- Value function:

$$v(s) = \sup_\sigma \inf_\tau \mathbb{E}_s \left[\mathbf{1}_{\{\sigma \leq \tau\}} L_\sigma + \mathbf{1}_{\{\tau < \sigma\}} U_\tau\right]$$
$$= \inf_\tau \sup_\sigma \mathbb{E}_s \left[\mathbf{1}_{\{\sigma \leq \tau\}} L_\sigma + \mathbf{1}_{\{\tau < \sigma\}} U_\tau\right]$$

- v_P value function of the American put, i.e.

$$v_P(s) = \sup_\sigma \mathbb{E}_s [L_\sigma]$$

- If $\delta \geq v_P(K)$ then game degenerates to Am. put (easy to see) $\Rightarrow$ assume $\delta \leq v_P(K)$

Theorem (Baurdoux & Kyprianou (2008))

- *A saddle point* (σ^*, τ^*) *exists, where*

$$\sigma^* = \inf\{t > 0 \,|\, S_t < s_h\} \text{ for some } s_h \in (0, K)$$
$$\tau^* = \inf\{t > 0 \,|\, S_t \in [K, s_w]\} \text{ for some } s_w \in [K, \infty)$$

- *Expressions for* s_h*, for* v *on* $(0, K]$ *& smooth pasting results.*

- *Question*: when $s_w = K$ and when $s_w > K$?
- *Answer*: depends on the structure of X
- *Idea of proof*: look at optimal response to the minimizer choosing $T_K := \inf\{t > 0 \,|\, S_t = K\}$, i.e.:

$$v_{max}(s) := \sup_{\sigma} \mathbb{E}_s \left[R_{\sigma, T_K}\right] = \mathbb{E}_s \left[R_{\sigma^*, T_K}\right]$$

  Then:
  - if $\forall s: \; v_{max}(s) \leq (K - s)^+ + \delta$ then $s_w = K$ (since $v \leq v_{max}$) and $v = v_{max}$
  - otherwise $v \neq v_{max} \Rightarrow s_w > K$
- *Next up*: graphs to illustrate 4 different cases. Red curve = v_{max}, Green curve = v when $s_w > K$

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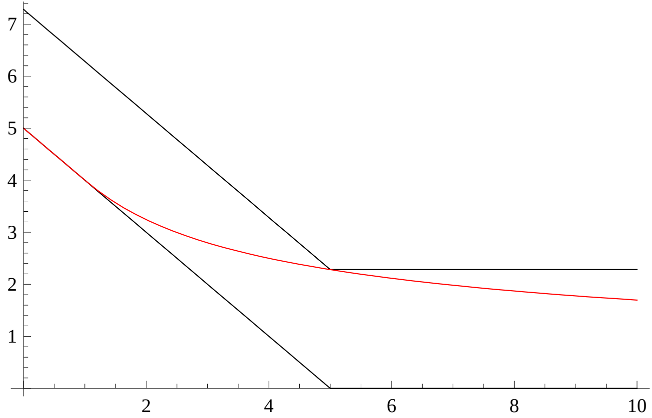
  Red curve = v_{max}, Green curve = v when $s_w > K$

Brownian motion case (Kyprianou (2004))

$\delta = v_P(K)$:



Brownian motion case (Kyprianou (2004))

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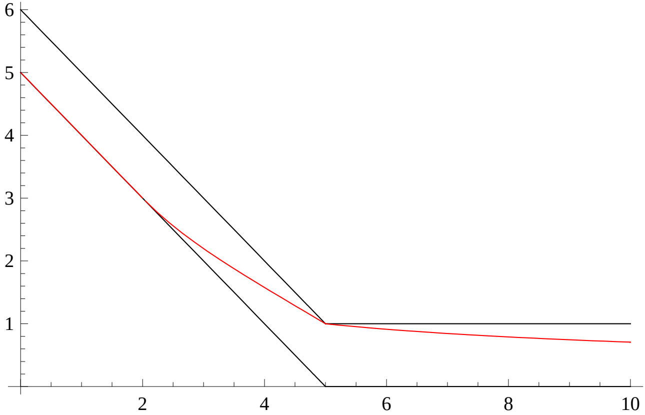

Bounded variation case (Baurd. & Kyp. (2008))

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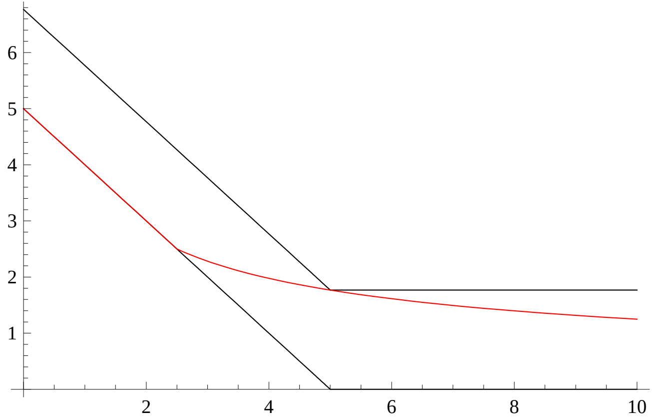

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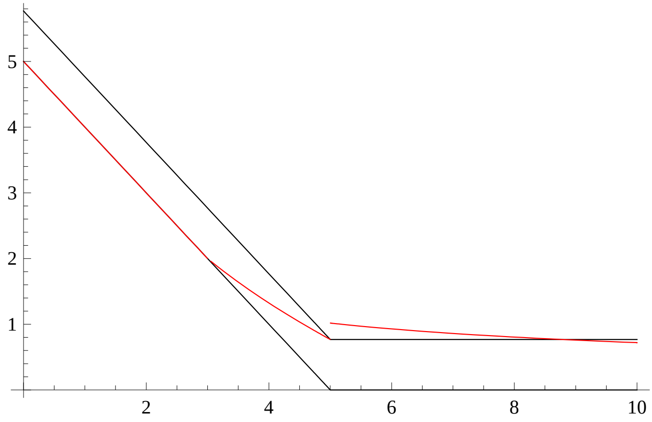
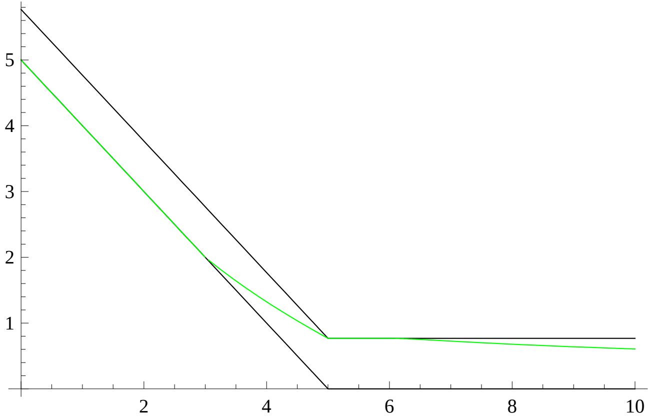


$$\Longrightarrow s_w > K$$

Bounded variation case (Baurd. & Kyp. (2008))

$\delta < v_P(K)$, actual value function:



Unb. var. case without Gaussian part (B. & K. (2008))

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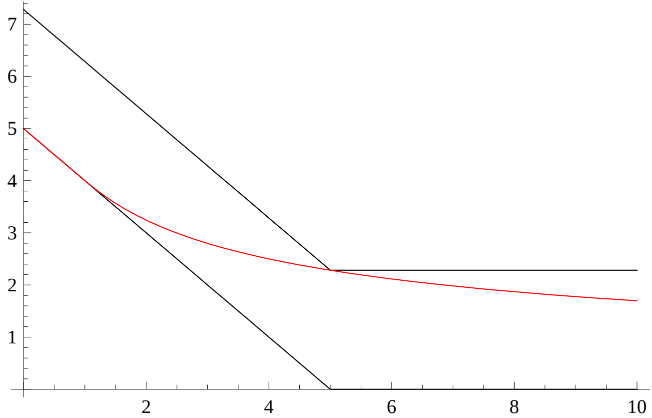

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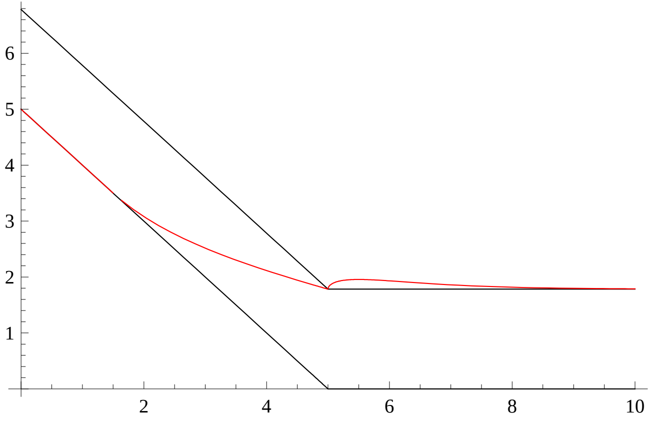

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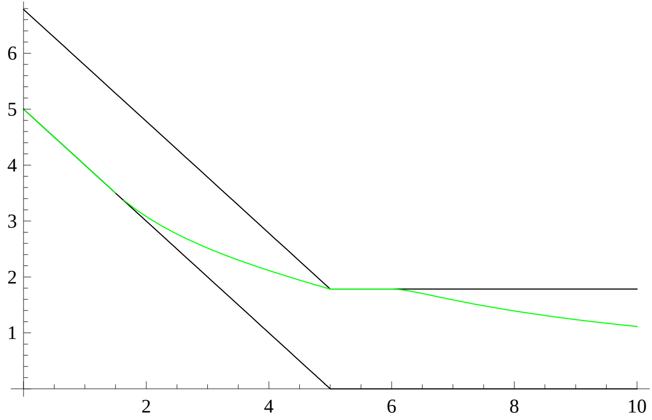

With Gaussian part (Baurdoux & KvS)

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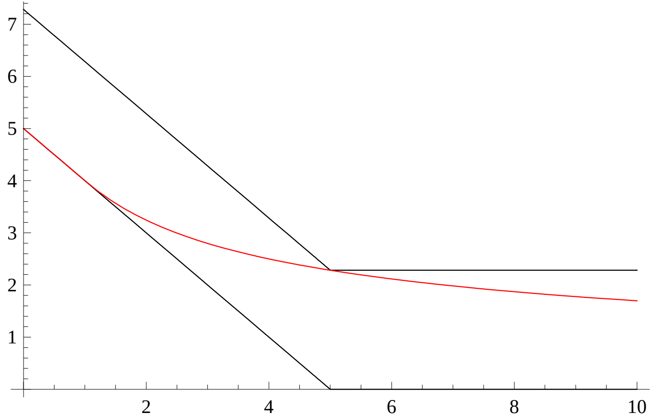
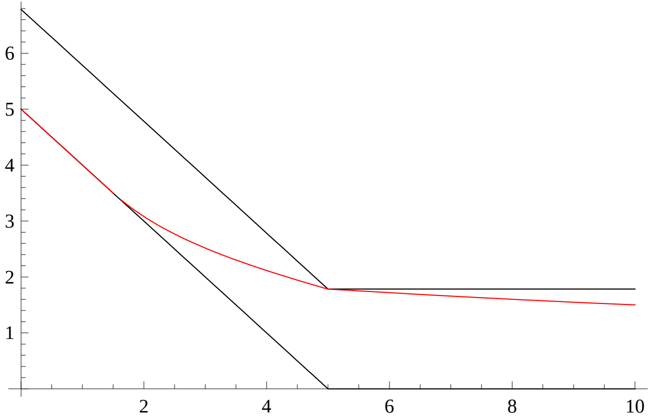


With Gaussian part (Baurdoux & KvS)

$\delta = v_P(K) - \varepsilon$:



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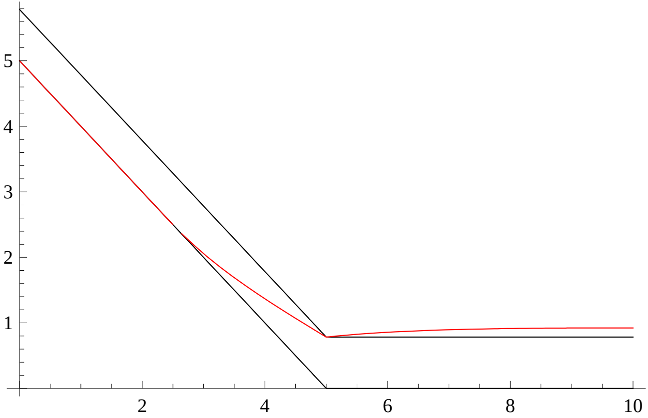
$\delta \ll v_P(K)$:



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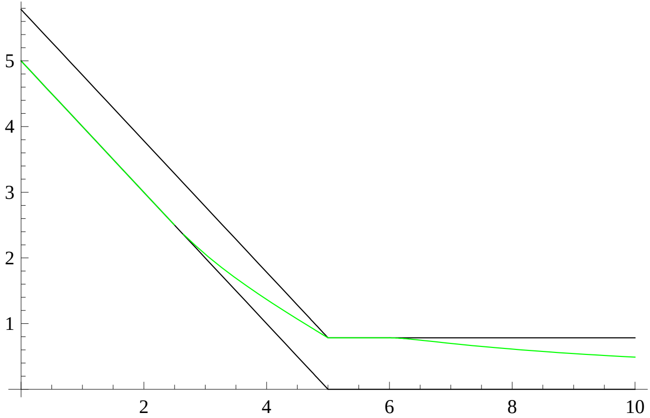

Theorem (Baurdoux & KvS (in progress))

If X has a positive Gaussian component and Lévy measure $\Pi \neq 0$, a threshold $\delta_0 \in (0, v_P(K))$ exists s.t.

- *if $\delta \in (0, \delta_0)$ then $s_w > K$*
- *if $\delta \in [\delta_0, v_P(K)]$ then $s_w = K$.*

Some additional results:

- $\delta \mapsto s_w(\delta)$ is cts. and decreasing
- $\lim_{\delta \uparrow \delta_0} s_w(\delta) = K$, $\lim_{\delta \downarrow 0} s_w(\delta) = K - \log(\inf \operatorname{supp}(\Pi))$
- expressions for s_w and v (more explicit in jump-diffusion case)

Finite expiry case ($T < \infty$), some examples

Numerical method: *Canadization* (Carr (1998)).

- discretise time stochastically, i.e. take grid points with distance iid exponentials $(\xi_i^{(n)})_{i\geq 1}$ with param. n, indep. of S
- define: $v_k^{(n)}$ = value of game with expiry date $\sum_{i=0}^{k} \xi_i^{(n)}$
- convergence: if $(k(n))$ s.t. $k(n)/n \to T$ as $n \to \infty$, then

$$\sum_{i=0}^{k(n)} \xi_i^{(n)} \xrightarrow{a.s.} T \quad \Longrightarrow \quad v_{k(n)}^{(n)}(s) \to v(T,s)$$

- algorithm for computing $(v_k^{(n)})_{k\geq 0}$: set $v_0^{(n)}(s) := (K-s)^+$. For $k \geq 1$, $v_k^{(n)}$ is a game with payoff:

$$\mathbf{1}_{\{\sigma\wedge\tau<\xi_n\}} R_{\sigma,\tau} + \mathbf{1}_{\{\sigma\wedge\tau\geq\xi_n\}} e^{-r\xi_n} v_{k-1}^{(n)}(S_{\xi_n}).$$

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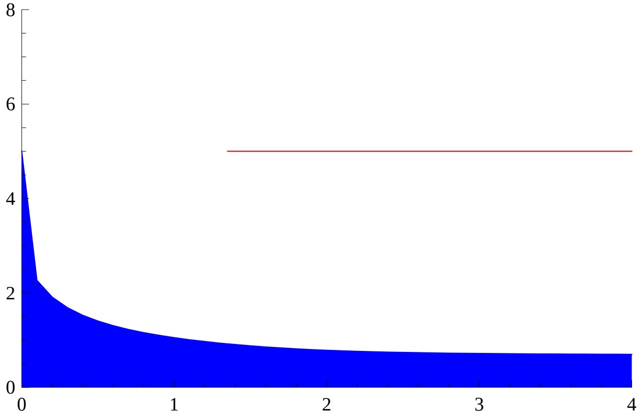
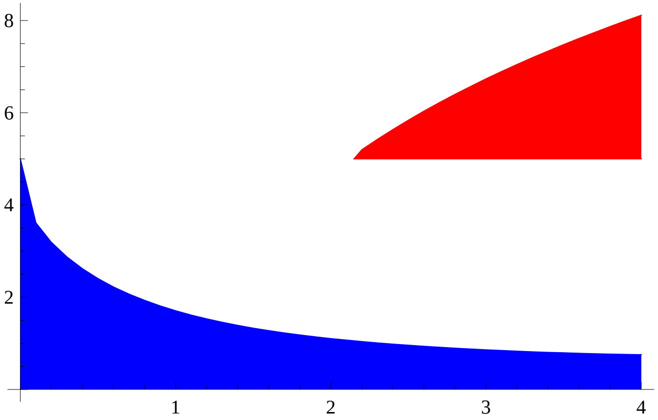
Brownian motion:



Jump-diffusion without Gaussian part:



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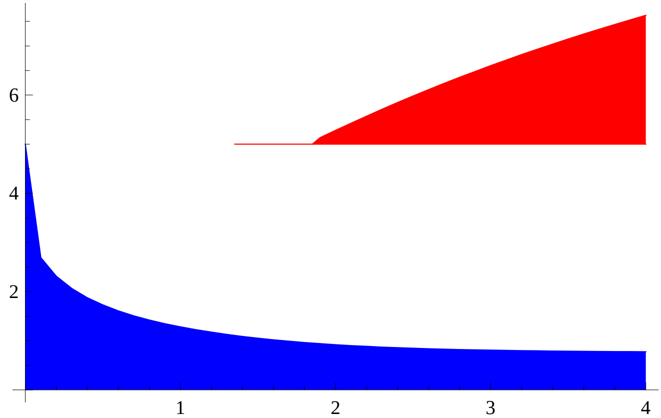

BAURDOUX, E.J. *and* KYPRIANOU, A.E. (2008). The McKean stochastic game driven by a spectrally negative Lévy process. *Electronic Journal of Probability* 8 (173–197).

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