Homework 2

1 Consider an upper half-plain $(y > 0)$ in $\mathbf{R}^2$ equipped with Riemannian metric

$$G = \sigma(x, y)(dx^2 + dy^2)\,,$$

a) Show that $\sigma > 0$,

b) In the case if $\sigma = \frac{1}{y^2}$ (the Lobachevsky metric) calculate the lengths of vectors $\mathbf{A} = 2\partial_x$ and $\mathbf{B} = 12\partial_x + 5\partial_y$ attached at the point $(x, y) = (1, 2)$,

c) calculate the cosine of the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$ and show that the answer does not depend on the choice of the function $\sigma(x, y)$,

d) Calculate the lenght of the segments $x = a+t, y = b$, and $x = a, y = b+t$, $0 \leq t \leq 1$ in the case if $\sigma = \frac{1}{y^2}$ (Lobachevsky plane)

e) Suppose $\sigma(x, y) = \frac{1}{(1+x^2+y^2)^2}$. Consider two curves L_1 and L_2 in upper half-plane such that

$$L_1 = \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \text{and } L_2 \begin{cases} x = g(t) \\ y = f(t) \end{cases}, \qquad 0 \leq t \leq 1\,,$$

where $f(t), g(t)$ are arbitrary functions $(f(t) > 0, g(t) > 0)$.

Show that these curves have the same length in the case if $\sigma(x, y) = \frac{1}{(1+x^2+y^2)^2}$.

2 a) Write down explicit formulae expressing stereographic coordinates for n-dimensional sphere $(x^1)^2 + \ldots + (x^{n+1})^2 = R^2$ of radius R via coordinates $x^1, \ldots, x^{n+1}$ and vice versa. (For simplicity you may consider cases $n = 1, 2, 3$.)

b)† Check that for unit sphere S^2, $(x^2 + y^2 + z^2 = 1)$ all the points with rational Cartesian coordinates x, y, z have rational stereographic coordinates u, v and vice versa.

3 Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

4 Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

5 Consider the surface L which is the upper sheet of two-sheeted hyperboloid in $\mathbf{R}^3$:

$$L: \qquad z^2 - x^2 - y^2 = 1, \quad z > 0\,,$$

a) Find parametric equation of the surface L using hyperbolic functions $\cosh, \sinh$ following an analogy with spherical coordinates on the sphere.

b) Consider the stereographic projection of the surface L on the plane OXY, i.e. the central projection on the plane $z = 0$ with the centre at the point $(0, 0, -1)$.

Show that the image of projection of the surface L is the open disc $x^2 + y^2 < 1$ in the plane OXY.

6* Consider the pseudo-Euclidean metric on $\mathbf{R}^3$ given by the formula

$$ds^2 = dx^2 + dy^2 - dz^2 \,. \tag{1}$$

Calculate the induced metric on the surface L considered in the Exercise 5, and show that it is a Riemannian metric (it is positive-definite).

Perform calculations in spherical-like coordinates (see Exercise 5a) above), and in stereographic coordinates (see exercise 5b) above).

Remark *The surface L sometimes is called pseudo-sphere. The Riemannian metric on this surface sometimes is called Lobachevsky (hyperbolic) metric. The surface L with this metric realises Lobachevsky (hyperbolic) geometry, where Euclid's 5-th Axiom fails. This Riemannian manifold (manifold+Riemannian metric) is called Lobachevsky (hyperbolic) plane. In stereographic coordinates Lobachevsky plane is realised as an open disc* $u^2 + v^2 < 1$ *in* $\mathbf{E}^2$. *It is so called Poincare model of Lobachevsky geometry. On the other hand Lobachevsky (hyperbolic plane) can be realised as a upper half-plane with metric*

$$G = \frac{dx^2 + dy^2}{y^2} \,, \tag{2}$$

(see the exercise 1 above). In the next exercise we will see how to compare these two realisations. of Lobachevsky plane.

7 * In the exercises 5 and 6 it was shown that pseudo-Euclidean metric (1) in $\mathbf{R}^3$ induces Riemanian metric on two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$. Show that it is not true for one-sheeted hyperboloid: metric on one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ in $\mathbf{R}^3$ is not Riemannian if it is induced with the pseudo-Euclidean metric (1).

8* In the exercise 6 we realised Lobachevsky plane as a disc $u^2 + v^2 < 1$. Find new coordinates x, y such that in these coordinates Lobachevsky plane (hyperbolic plane) can be considered as an upper half plane $\{(x, y) \colon y > 0\}$ in $\mathbf{E}^2$ and write down explicitly Riemannian metric in these coordinates.

Hint: *You may use complex coordinates:* $z = x + iy, \bar{z} = x - iy, \omega = u + iv, \bar{w} = u - iv$, *and consider a holomorphic transformation:* $\omega = \frac{1+iz}{1-iz} \Leftrightarrow z = i\frac{1-\omega}{1+\omega}$, *which transforms the open disc* $w\bar{w} < 1$ *onto the upper plane* $\mathbf{Im} z > 0$.

Remark Later by default we will call "Lobachevsky (hyperbolic) plane" the realisation of Lobachevsky plane as an half-upper plane in $\mathbf{E}^2$ with coordinates x, y $(y > 0)$. (Riemannian metric is given by equation (2)).