Homework 4. Solutions

1 Calculate the Christoffel symbols of the canonical flat connection in $E^3$ in
a) cylindrical coordinates $(x = r \cos \varphi, y = r \sin \varphi, z = h)$,
 b) spherical coordinates.

(For the case of sphere try to make calculations at least for components $\Gamma^r_{rr}, \Gamma^r_{r\varphi}, \Gamma^r_{\varphi\varphi}, \Gamma^r_{\theta\theta}, \ldots, \Gamma^r_{\varphi\varphi}$)

Remark One can calculate Christoffel symbols using Levi-Civita Theorem. There is a third way to
calculate Christoffel symbols: It using approach of Lagrangian. This is the easiest way. (see the Homework
6)

In cylindrical coordinates $(r, \varphi, h)$ we have

$$\begin{align*}
(x &= r \cos \varphi \\
y &= r \sin \varphi \\
z &= h)
\end{align*}$$

and

$$\begin{align*}
(r &= \sqrt{x^2 + y^2} \\
\varphi &= \arctan \frac{y}{x} \\
h &= z)
\end{align*}$$

We know that in Cartesian coordinates all Christoffel symbols vanish. Hence in cylindrical coordinates
(see in detail lecture notes):

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial r \partial r} + \frac{\partial^2 y}{\partial r \partial r} + \frac{\partial^2 z}{\partial r \partial z} = 0,$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial \varphi \partial r} + \frac{\partial^2 y}{\partial \varphi \partial r} + \frac{\partial^2 z}{\partial \varphi \partial z} = -\sin \varphi \cos \varphi + \sin \varphi \cos \varphi = 0.$$

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi^2} + \frac{\partial^2 y}{\partial \varphi^2} + \frac{\partial^2 z}{\partial \varphi^2} = -\frac{x}{r} - \frac{y}{r} = -r.$$

$$\Gamma^r_{\theta\theta} = \frac{\partial^2 x}{\partial \theta^2} + \frac{\partial^2 y}{\partial \theta^2} + \frac{\partial^2 z}{\partial \theta^2} = 0.$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\partial^2 x}{\partial \varphi \partial r} + \frac{\partial^2 y}{\partial \varphi \partial r} + \frac{\partial^2 z}{\partial \varphi \partial z} = -\sin \varphi \cos \varphi + \sin \varphi \cos \varphi = 0.$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi^2} + \frac{\partial^2 y}{\partial \varphi^2} + \frac{\partial^2 z}{\partial \varphi^2} = -\frac{x}{r} - \frac{y}{r} = 0.$$

All symbols $\Gamma^r_{hh}, \Gamma^\varphi_{hh}$ vanish

$$\Gamma^r_{rr} = \Gamma^r_{hh} = \Gamma^r_{\varphi h} = \Gamma^r_{h \varphi} = \Gamma^r_{hh} = dots = 00$$

since $\frac{\partial^2 x}{\partial \varphi \partial \varphi} = \frac{\partial^2 y}{\partial \varphi \partial \varphi} = \frac{\partial^2 z}{\partial \varphi \partial \varphi} = 0$

For all symbols $\Gamma^h_{hh}, \Gamma^h_{hh}$ since $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$ and $\frac{\partial h}{\partial z} = 1$. On the other hand all $\frac{\partial^2 z}{\partial \varphi \partial \varphi}$ vanish. Hence
all symbols $\Gamma^h_{hh}$ vanish.

b) spherical coordinates

$$\begin{align*}
x &= r \sin \varphi \\
y &= r \sin \varphi \\
z &= r \cos \theta
\end{align*}$$

$$\begin{align*}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\
\varphi &= \arctan \frac{y}{x}
\end{align*}$$

We already know the fast way to calculate Christofel symbol using Lagrangian of free particle and this
method work for a flat connection since flat connection is a Levi-Civita connection for Euclidean metric.
So perform now brute force calculations only for some components. (Then later (in homework 6) we
will calculate using very quickly Lagrangian of free particle.)

\[ \Gamma_{rv} = 0 \text{ since } \frac{\partial^2 x^i}{\partial x^r \partial x^v} = 0. \]

\[ \Gamma_{r\theta} = \Gamma_{\theta r} = \frac{\partial^2 x^i}{\partial r \partial \theta} + \frac{\partial^2 y}{\partial r \partial \theta} + \frac{\partial^2 z}{\partial r \partial \theta} = \cos \theta \cos \frac{x}{r} + \cos \theta \sin \frac{y}{r} - \sin \frac{z}{r} = 0, \]

\[ \Gamma_{\theta \phi} = \Gamma_{\phi \theta} = -r \sin \theta \cos \frac{x}{r} - r \sin \theta \sin \frac{y}{r} - r \cos \frac{z}{r} = -r \]

\[ \Gamma_{r \phi} = \Gamma_{\phi r} = \frac{\partial^2 x}{\partial r \partial \phi} + \frac{\partial^2 y}{\partial r \partial \phi} + \frac{\partial^2 z}{\partial r \partial \phi} = -\sin \theta \sin \frac{x}{r} + \sin \theta \cos \frac{y}{r} = 0 \]

and so on....

2 a) Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system:

\[ \Gamma_{km}^i = \Gamma_{mk}^i. \]

Show that they are symmetric in an arbitrary coordinate system.

b) Show that the Christoffel symbols of connection \( \nabla \) are symmetric (in any coordinate system) if and only if

\[ \nabla_X Y - \nabla_Y X - [X, Y] = 0, \]

for arbitrary vector fields \( X, Y \).

c) Consider for an arbitrary connection the following operation on the vector fields:

\[ S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \]

and find its properties.

Solution

a) Let \( \Gamma_{k}^{i} = \Gamma_{m}^{i} \). We have to prove that \( \Gamma_{km}^{i} = \Gamma_{mk}^{i} \).

We have

\[ \Gamma_{km}^{i} = \frac{\partial x^{i}}{\partial x^{m}} \frac{\partial x^{k}}{\partial x^{m}} \Gamma_{km}^{i} = \frac{\partial x^{i}}{\partial x^{m}} \frac{\partial x^{k}}{\partial x^{m}} \Gamma_{km}^{i} + \frac{\partial x^{i}}{\partial x^{m}} \frac{\partial x^{k}}{\partial x^{m}} \Gamma_{km}^{i} \]

Hence

\[ \Gamma_{km}^{i} = \frac{\partial x^{i}}{\partial x^{m}} \frac{\partial x^{k}}{\partial x^{m}} \Gamma_{km}^{i} + \frac{\partial x^{i}}{\partial x^{m}} \frac{\partial x^{k}}{\partial x^{m}} \Gamma_{km}^{i} \]

But \( \Gamma_{km}^{i} = \Gamma_{mk}^{i} \) and \( \frac{\partial x^{i}}{\partial x^{m}} \frac{\partial x^{k}}{\partial x^{m}} = \frac{\partial x^{i}}{\partial x^{k}} \frac{\partial x^{m}}{\partial x^{m}} \).

Hence

\[ \Gamma_{km}^{i} = \frac{\partial x^{i}}{\partial x^{m}} \frac{\partial x^{k}}{\partial x^{m}} \Gamma_{km}^{i} + \frac{\partial x^{i}}{\partial x^{m}} \frac{\partial x^{k}}{\partial x^{m}} \Gamma_{km}^{i} \]

b) The relation

\[ \nabla_X Y - \nabla_Y X - [X, Y] = 0 \]

holds for all fields if and only if it holds for all basic fields. One can easy check it using axioms of connection (see the next part). Consider \( X = \frac{\partial}{\partial x^{i}}, Y = \frac{\partial}{\partial x^{j}} \) then since \( [\partial_i, \partial_j] = 0 \) we have that

\[ \nabla_X Y - \nabla_Y X - [X, Y] = \nabla_i \partial_j - \nabla_j \partial_i = \Gamma_{ij}^k \partial_k - \Gamma_{ji}^k \partial_k = (\Gamma_{ij}^k - \Gamma_{ji}^k) \partial_k = 0 \]

We see that commutator for basic fields \( \nabla_X Y - \nabla_Y X - [X, Y] = 0 \) if and only if \( \Gamma_{ij}^k - \Gamma_{ji}^k = 0 \).

c) One can easy check it by straightforward calculations or using axioms for connection that \( S(X, Y) \) is a vector-valued bilinear form on vectors. In particularly \( S(fX, Y) = fS(X, Y) \) for an arbitrary (smooth) function. Show this just using axioms defining connection:

\[ S(fX, Y) = \nabla fX Y - \nabla Y (fX) - [fX, Y] = f \nabla X Y - f \nabla Y X - \partial_X f X + [Y, f X] = \]

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\[ f \nabla_Y X = f \nabla_Y X - (\partial_Y f) X + \partial_Y f X + f [Y, X] = f(\nabla_X Y - \nabla_Y X - [X, Y]) = f S(X, Y) \]

3 Let \( \nabla_1, \nabla_2 \) be two different connections. Let \( \Gamma_{km}^i \) and \( \tilde{\Gamma}_{km}^i \) be the Christoffel symbols of connections \( \nabla_1 \) and \( \nabla_2 \) respectively.

a) Find the transformation law for the object: \( T^i_{km} = \Gamma_{km}^i - \tilde{\Gamma}_{km}^i \) under a change of coordinates. 

Show that it is \( \left( \frac{1}{2} \right) \) tensor.

b) Consider an operation \( \nabla_1 - \nabla_2 \) on vector fields and find its properties.

Christoffel symbols of both connections transform according the law (1). The second term is the same. Hence it vanishes for their difference:

\[ T^i_{km'} = \Gamma_{km'}^i - \Gamma_{km'}^i = \frac{\partial x^i}{\partial x'^{m'}} \frac{\partial x'^{m'}}{\partial x^m} \Gamma_{km}^i - \frac{\partial x^i}{\partial x'^{m'}} \frac{\partial x'^{m'}}{\partial x^m} \tilde{\Gamma}_{km}^i \]

We see that \( T^i_{km'} \) transforms as a tensor of the type \( \left( \frac{1}{2} \right) \).

b) One can do it in invariant way. Using axioms of connection study \( T = \nabla_1 - \nabla_2 \) is a vector field. Consider

\[ T(X, Y) = \nabla_1 X Y - \nabla_2 X Y \]

Show that \( T(fX, Y) = fT(X, Y) \) for an arbitrary (smooth) function, i.e. it does not possesses derivatives:

\[ T(fX, Y) = \nabla_1 fX Y - \nabla_2 fX Y = (\partial_X f) Y + f \nabla_1 X Y - \nabla_1 fX Y - \nabla_2 fX Y = fT(X, Y). \]

4 a) Consider \( t_m = \Gamma_{km}^i \). Show that the transformation law for \( t_m \) is

\[ t_{m'} = \frac{\partial x^m}{\partial x'^n} t_m + \frac{\partial^2 x^r}{\partial x'^m \partial x'^n} \frac{\partial x^m}{\partial x'^r}. \]

b) Show that this law can be written as

\[ t_{m'} = \frac{\partial x^m}{\partial x'^n} t_m + \frac{\partial}{\partial x'^n} \left( \log \det \left( \frac{\partial x}{\partial x'} \right) \right). \]

Solution. Using transformation law (1) we have

\[ t_{m'} = \Gamma_{i'm'}^i = \frac{\partial x^i}{\partial x'^i} \frac{\partial x'k}{\partial x'^m} \frac{\partial x'^m}{\partial x^k} \Gamma_{km}^i + \frac{\partial x^r}{\partial x'^i} \frac{\partial x'^m}{\partial x^r} \Gamma_{km}^i. \]

We have that \( \frac{\partial x^i}{\partial x'^i} \frac{\partial x'k}{\partial x'^m} \frac{\partial x'^m}{\partial x^k} = \delta_i^k \). Hence

\[ t_{m'} = \Gamma_{i'm'}^i = \frac{\partial x^i}{\partial x'^i} \frac{\partial x'k}{\partial x'^m} \frac{\partial x'^m}{\partial x^k} \Gamma_{km}^i + \frac{\partial x^r}{\partial x'^i} \frac{\partial x'^m}{\partial x^r} \Gamma_{km}^i = \delta_i^k \frac{\partial x^m}{\partial x^k} \Gamma_{km}^i + \frac{\partial x^r}{\partial x'^i} \frac{\partial x'^m}{\partial x^r} \Gamma_{km}^i = \delta_i^k \Gamma_{km}^i + \frac{\partial x^r}{\partial x'^i} \frac{\partial x'^m}{\partial x^r} \Gamma_{km}^i. \]

b) When calculating \( \frac{\partial}{\partial x'^m} \left( \log \det \left( \frac{\partial x}{\partial x'} \right) \right) \) use very important formula:

\[ \delta \det A = \det A \text{Tr}(A^{-1} \delta A) \rightarrow \delta \log \det A = \text{Tr}(A^{-1} \delta A). \]

Hence

\[ \frac{\partial}{\partial x'^m} \left( \log \det \left( \frac{\partial x}{\partial x'} \right) \right) = \frac{\partial x^i}{\partial x'^i} \frac{\partial^2 x^r}{\partial x'^i \partial x'^m}. \]
and we come to transformation law for (1).

To deduce the formula for \( \delta \det A \) notice that

\[
\det(A + \delta A) = \det A \det(1 + A^{-1} \delta A)
\]

and use the relation: \( \det(1 + \delta A) = 1 + \text{Tr} \delta A + O(\delta^2 A) \)

5 Calculate Christoffel symbols of the connection induced on the surface \( M \) in \( \mathbb{E}^n \) equipped with canonical flat connection.

a) \( M = S^1 \) in \( \mathbb{E}^2 \)
b) \( M - \) parabola \( y = x^2 \) in \( \mathbb{E}^2 \)
c) \( M - \) cylinder, cone, sphere in \( \mathbb{E}^3 \).
d) saddle \( z = xy \)

Solution.

a) Consider polar coordinate on \( S^1 \), \( x = R \cos \varphi, y = R \sin \varphi \). We have to define the connection on \( S^1 \) induced by the canonical flat connection on \( \mathbb{E}^2 \). It suffices to define \( \nabla \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} = \Gamma_{\varphi \varphi \varphi} \frac{\partial}{\partial \varphi} \).

Recall the general rule. Let \( r(u^\alpha) \): \( x_i = x_i(u^\alpha) \) is embedded surface in Euclidean space \( \mathbb{E}^n \). The basic vectors \( \frac{\partial}{\partial u^\alpha} = \frac{\partial r(u^\alpha)}{\partial u^\alpha} \). To take the induced covariant derivative \( \nabla_X Y \) for two tangent vectors \( X, Y \) we take a usual derivative of vector \( Y \) along vector \( X \) (the derivative with respect to canonical flat connection: in Cartesian coordiantes is just usual derivatives of components) then we take the tangent component of the answer, since in general derivative of vector \( Y \) along vector \( X \) is not tangent to surface:

\[
\nabla_X \frac{\partial}{\partial u^\alpha} = \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial u^\gamma} = \left( \nabla_{\frac{\partial}{\partial u^\alpha}} \frac{\partial}{\partial u^\gamma} \right)_{\text{tangent}} = \left( \frac{\partial^2 r(u)}{\partial u^\alpha \partial u^\gamma} \right)_{\text{tangent}}
\]

\((\nabla_{\text{canonical}} \frac{\partial}{\partial u^\alpha} \frac{\partial}{\partial u^\beta})\) is just usual derivative in Euclidean space since for canonical connection all Christoffel symbols vanish.)

In the case of 1-dimensional manifold, curve it is just tangential acceleration!:

\[
\nabla_X \frac{\partial}{\partial u} = \Gamma^u_{\alpha\beta} \frac{\partial}{\partial u} = \left( \nabla_{\frac{\partial}{\partial u^\alpha}} \frac{\partial}{\partial u} \right)_{\text{tangent}} = \left( \frac{\partial^2 r(u)}{du^\alpha du^\gamma} \right)_{\text{tangent}} = \mathbf{a}_{\text{tangent}}
\]

For the circle \( S^1 \), \( (x = R \cos \varphi, y = R \sin \varphi) \), in \( \mathbb{E}^2 \). We have

\[
\mathbf{r}_\varphi = \frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi} \frac{\partial}{\partial y} = -R \sin \varphi \frac{\partial}{\partial x} + R \cos \varphi \frac{\partial}{\partial y},
\]

\[
\nabla_X \frac{\partial}{\partial \varphi} = \Gamma^\varphi_{\varphi \varphi} \frac{\partial}{\partial \varphi} = \left( \nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial \varphi} \right)_{\text{tangent}} = \left( \frac{\partial^2 \mathbf{r}(u)}{\partial \varphi^2} \right)_{\text{tangent}} = 0,
\]

since the vector \(-R \cos \varphi \frac{\partial}{\partial x} - R \sin \varphi \frac{\partial}{\partial y}\) is orthogonal to the tangent vector \( \mathbf{r}_\varphi \). In other words it means that acceleration is centripetal: tangential acceleration equals to zero.

We see that in coordinate \( \varphi \), \( \Gamma^\varphi_{\varphi \varphi} = 0 \).

Additional work: Perform calculation of Christoffel symbol in stereographic coordinate \( t \):

\[
x = \frac{2tR^2}{R^2 + t^2}, \ y = \frac{R((t^2 - R^2)}{t^2 + R^2}.
\]
In this case
\[ \mathbf{r}_t = \frac{\partial}{\partial t} = \frac{\partial x}{\partial t} \frac{\partial}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial}{\partial y} = \frac{2R^2}{(R^2 + t^2)^2} \left( (R^2 - t^2) \frac{\partial}{\partial x} + 2tR \frac{\partial}{\partial y} \right), \]
\[ \nabla \frac{\partial}{\partial t} = \Gamma^r_{tt} \partial_t = \left( \nabla^{(\text{canonic})}_{\partial_r} \right)_{\text{tangent}} \left( \frac{\partial}{\partial t} \mathbf{r}_t \right)_{\text{tangent}} = (r_{tt})_{\text{tangent}} = \left( -\frac{4t}{t^2 + R^2} \mathbf{r}_t + \frac{2R^2}{(R^2 + t^2)^2} \left( -2t \frac{\partial}{\partial x} + 2R \frac{\partial}{\partial y} \right) \right)_{\text{tangent}} \]

We have
\[ n_t = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{R^2 + t^2} \left( 2tR \partial_x + (t^2 - R^2) \partial_y \right), \]
where \( \langle \mathbf{r}_t, n_t \rangle = 0 \). Hence \( \langle \mathbf{r}_t, n_t \rangle = -\frac{2R^3}{(t^2 + R^2)^2} \) and
\[ (r_{tt})_{\text{tangent}} = \mathbf{r}_t - \langle \mathbf{r}_t, n_t \rangle n = \left( -\frac{4t}{t^2 + R^2} \mathbf{r}_t + \frac{2R^2}{(R^2 + t^2)^2} \left( -2t \frac{\partial}{\partial x} + 2R \frac{\partial}{\partial y} \right) \right)_{\text{tangent}} \]

We come to the answer:
\[ \nabla \partial_t = \frac{\langle \partial_t, \partial_r \rangle}{\langle \partial_t, n \rangle} \frac{\partial_r}{\partial_t} = \frac{\langle \partial_t, \partial_r \rangle}{\langle \partial_t, n \rangle} \Gamma^{r}_{tt}, \quad \text{i.e.} \Gamma^{r}_{tt} = \frac{-2t}{t^2 + R^2} \]

Of course we could calculate the Christoffel symbol in stereographic coordinates just using the fact that we already know the Christoffel symbol in polar coordinates: \( \Gamma^{\varphi}_{\varphi \varphi} = 0 \), hence
\[ \Gamma^{r}_{tt} = \frac{dt}{d\varphi} \frac{d\varphi}{dx} \frac{d\varphi}{dx} \Gamma^{\varphi}_{\varphi \varphi} + \frac{dt}{d\varphi} \frac{d\varphi}{dx} \frac{d\varphi}{dx} \frac{dt}{d\varphi} = \frac{d^2 \varphi}{dt^2} \frac{dt}{d\varphi} \]

It is easy to see that \( t = R \tan \left( \frac{\varphi}{2} \right) \), i.e. \( \varphi = 2 \arctan \frac{t}{R} - \frac{\pi}{2} \) and
\[ \Gamma^{r}_{tt} = \frac{d^2 \varphi}{dt^2} \frac{dt}{d\varphi} = \frac{d\varphi}{dx} \frac{dt}{d\varphi} = -\frac{2t}{t^2 + R^2}. \]

b) For parabola \( x = t, y = t^2 \)
\[ \mathbf{r}_t = \frac{\partial}{\partial t} = \frac{\partial x}{\partial t} \frac{\partial}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial}{\partial y} = \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial y}, \]
\[ \nabla \frac{\partial}{\partial t} = \Gamma^{r}_{tt} \partial_t = \left( \nabla^{(\text{canonic})}_{\partial_r} \right)_{\text{tangent}} \left( \frac{\partial}{\partial t} \mathbf{r}_t \right)_{\text{tangent}} = (r_{tt})_{\text{tangent}} = \left( 2 \frac{\partial}{\partial y} \right)_{\text{tangent}} \]

To calculate \( (r_{tt})_{\text{tangent}} \) we need to extract its orthogonal component: \( (r_{tt})_{\text{tangent}} = \mathbf{r}_t - \langle \mathbf{r}_tt, n_t \rangle n \), where \( n \) is an orthogonal unit vector: \( \langle n, r_t \rangle = 0, \langle n, n \rangle = 1 \):
\[ n_t = \frac{1}{\sqrt{1 + 4t^2}} \left( -2t \partial_x + \partial_y \right). \]

We have
\[ (r_{tt})_{\text{tangent}} = \mathbf{r}_t - \langle \mathbf{r}_tt, n_t \rangle n = 2 \partial_y - \frac{1}{\sqrt{1 + 4t^2}} \left( -2t \partial_x + \partial_y \right) \]
\[ \frac{1}{\sqrt{1 + 4t^2}} \left( -2t \partial_x + \partial_y \right) = \]
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\[ \frac{4t}{1 + 4t^2} \partial_x + \frac{8t^2}{1 + 4t^2} \partial_y = \frac{4t}{1 + 4t^2} (\partial_x + 2t \partial_y) = \frac{4t}{1 + 4t^2} \partial_t \]

We come to the answer:
\[ \nabla \partial_t \partial_t = \frac{4t}{1 + 4t^2} \partial_t, \quad \text{i.e.} \Gamma^t_{tt} = \frac{4t}{1 + 4t^2} \]

**Remark** Do not be surprised by resemblance of the answer to the answer for circle in stereographic coordinates.

c) cylinder, cone and sphere

a) Cylinder

\[ r(h, \varphi) = \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \\ z = h \end{cases} \]

\[ \partial_h = r_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \partial_\varphi = r_\varphi = \begin{pmatrix} -a \sin \varphi \\ a \cos \varphi \\ 0 \end{pmatrix} \]

Calculate
\[ \nabla \partial_h \partial_h = \Gamma^h_{hh} \partial_h + \Gamma^\varphi_{hh} \partial_\varphi = \left( \frac{\partial^2 r}{\partial h \partial \varphi} \right)_{\text{tangent}} = 0 \text{ since } r_{hh} = 0. \]

Hence \( \Gamma^h_{hh} = \Gamma^\varphi_{hh} = 0 \)

\[ \nabla \partial_\varphi \partial_\varphi = \Gamma^h_{\varphi \varphi} \partial_h + \Gamma^\varphi_{\varphi \varphi} \partial_\varphi = \left( \frac{\partial^2 r}{\partial \varphi \partial \varphi} \right)_{\text{tangent}} = \left( \begin{array}{c} -a \cos \varphi \\ -a \sin \varphi \\ 0 \end{array} \right)_{\text{tangent}} = 0 \]

since the vector \( r_{\varphi \varphi} = \left( \begin{array}{c} -a \cos \varphi \\ -a \sin \varphi \\ 0 \end{array} \right) \) is orthogonal to the surface of cylinder. Hence \( \Gamma^h_{h \varphi} = \Gamma^h_{\varphi h} = \Gamma^\varphi_{h \varphi} = \Gamma^\varphi_{\varphi h} = 0 \).

We see that for cylinder all Christoffel symbols in cylindrical coordinates vanish. This is not big surprise: in cylindrical coordinates metric equals \( dh^2 = a^2 d\varphi^2 \). This due to Levi-Civita theorem one can see that Levi-Civita which equals to induced connection vanishes since all coefficients are constants.

For cone: see Coursework problem 3.

For the sphere \( r(\theta, \varphi) = \begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases} \), we have

\[ \frac{\partial}{\partial \theta} = r_\theta = \begin{pmatrix} R \cos \theta \cos \varphi \\ R \cos \theta \sin \varphi \\ -R \sin \theta \end{pmatrix}, \quad \frac{\partial}{\partial \varphi} = r_\varphi = \begin{pmatrix} -R \sin \theta \sin \varphi \\ R \sin \theta \cos \varphi \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} R \sin \theta \cos \varphi \\ R \sin \theta \sin \varphi \\ R \cos \theta \end{pmatrix} \]

Calculate
\[ \nabla \partial_\theta \partial_\theta = \Gamma^\theta_{\theta \theta} \partial_\theta + \Gamma^\varphi_{\theta \varphi} \partial_\varphi = \left( \frac{\partial^2 r}{\partial \theta^2} \right)_{\text{tangent}} = 0 \]
since $\frac{\partial^2 r}{\partial \varphi^2} = -Rn$ is orthogonal to the sphere. Hence $\Gamma^{\theta}_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = 0$.

Now calculate

$$\nabla_{\partial_\varphi} \partial_\varphi = \Gamma^\theta_{\theta\varphi} \partial_\theta + \Gamma^\varphi_{\varphi\varphi} \partial_\varphi = \left( \frac{\partial^2 r}{\partial \theta \partial \varphi} \right)_{\text{tangent}}.$$  

We have

$$\frac{\partial^2 r}{\partial \theta \partial \varphi} = \cot \theta r_{\varphi},$$

hence

$$\nabla_{\partial_\varphi} \partial_\varphi = \Gamma^\theta_{\theta\varphi} \partial_\theta + \Gamma^\varphi_{\varphi\varphi} \partial_\varphi = \left( \frac{\partial^2 r}{\partial \theta \partial \varphi} \right)_{\text{tangent}} = \cot \theta r_{\varphi}, \text{ i.e.}$$

$\Gamma^\theta_{\theta\varphi} = 0, \Gamma^\varphi_{\varphi\varphi} = \cot \theta$.

Now calculate

$$\nabla_{\partial_\theta} \partial_\varphi = \Gamma^\theta_{\theta\varphi} \partial_\theta + \Gamma^\varphi_{\varphi\varphi} \partial_\varphi = \left( \frac{\partial^2 r}{\partial \theta \partial \varphi} \right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2 r}{\partial \theta^2} = \cot \theta r_{\varphi},$$

hence

$$\nabla_{\partial_\theta} \partial_\varphi = \Gamma^\theta_{\theta\varphi} \partial_\theta + \Gamma^\varphi_{\varphi\varphi} \partial_\varphi = \left( \frac{\partial^2 r}{\partial \theta \partial \varphi} \right)_{\text{tangent}} = \cot \theta r_{\varphi}, \text{ i.e.}$$

$\Gamma^\theta_{\theta\varphi} = 0, \Gamma^\varphi_{\varphi\varphi} = \cot \theta$. Of course we did not need to perform these calculations: since $\nabla$ is symmetric connection and $\nabla_{\partial_\theta} \partial_\varphi = \nabla_{\partial_\varphi} \partial_\theta$, i.e.

$$\Gamma^\theta_{\varphi\theta} = \Gamma^\theta_{\theta\varphi} = 0 \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{\varphi\varphi} = \cot \theta.$$

and finally

$$\nabla_{\partial_\varphi} \partial_\varphi = \Gamma^\theta_{\theta\varphi} \partial_\theta + \Gamma^\varphi_{\varphi\varphi} \partial_\varphi = \left( \frac{\partial^2 r}{\partial \varphi^2} \right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2 r}{\partial \varphi^2} = \begin{pmatrix} -R \sin \theta \cos \varphi \\ -R \sin \theta \sin \varphi \\ 0 \end{pmatrix} = -\sin \theta \cos \theta \begin{pmatrix} R \cos \theta \cos \varphi \\ R \cos \theta \sin \varphi \\ -R \sin \theta \end{pmatrix} - R \sin^2 \theta \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix} = -\sin \theta \cos \theta r_\theta - R \sin^2 \theta n,$$

hence

$$\nabla_{\partial_\varphi} \partial_\varphi = \Gamma^\theta_{\theta\varphi} \partial_\theta + \Gamma^\varphi_{\varphi\varphi} \partial_\varphi = \left( \frac{\partial^2 r}{\partial \varphi^2} \right)_{\text{tangent}} = -\sin \theta \cos \theta r_\theta, \text{ i.e.}$$

$\Gamma^\theta_{\varphi\theta} = \sin \theta \cos \theta, \Gamma^\varphi_{\varphi\varphi} = 0$.

For saddle $z = xy$: We have $r(u, v)$:

$$\begin{cases} x = u \\ y = v \\ z = uv \end{cases}, \quad \partial_u = r_u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \partial_v = r_v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

It will be useful also to use the normal unit vector $n = \frac{1}{\sqrt{1 + u^2 + v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}$.

Calculate:

$$\nabla_{\partial_u} \partial_u = \Gamma^u_{uu} \partial_u + \Gamma^v_{uv} \partial_v = \left( \frac{\partial^2 r}{\partial u^2} \right)_{\text{tangent}} = (r_{uu})_{\text{tangent}} = 0 \text{ since } r_{uu} = 0.$$
Hence $\Gamma^u_{uu} = \Gamma^v_{uu} = 0$.
Analogously $\Gamma^u_{vv} = \Gamma^v_{vv} = 0$ since $r_{vv} = 0$.
Now calculate $\Gamma^u_{uv}, \Gamma^v_{uv}, \Gamma^u_{vu}, \Gamma^v_{vu}$:

$$\nabla_{\partial_u} \partial_v = \nabla_{\partial_v} \partial_u = \Gamma^w_{uv} \partial_u + \Gamma^w_{vu} \partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}}$$

Using normal unit vector $\mathbf{n}$ we have: $(\mathbf{r}_{uv})_{\text{tangent}} = \mathbf{r}_{uv} - (\mathbf{r}_{uv}, \mathbf{n}) \mathbf{n} = \Gamma^u_{uv} \partial_u + \Gamma^v_{uv} \partial_v = 

\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}} - \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{1 + u^2 + v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{1 + u^2 + v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} = 

\frac{1}{1 + u^2 + v^2} \begin{pmatrix} v \\ u \\ u^2 + v^2 \end{pmatrix} = \frac{v}{1 + u^2 + v^2} \begin{pmatrix} 0 \\ 1 \\ v \end{pmatrix} + \frac{u}{1 + u^2 + v^2} \begin{pmatrix} 0 \\ u \\ v \end{pmatrix} = \frac{v \mathbf{n} + u \mathbf{r}_v}{1 + u^2 + v^2}.$$

Hence $\Gamma^u_{uv} = \Gamma^v_{uv} = \frac{v}{1 + u^2 + v^2}$ and $\Gamma^u_{vv} = \Gamma^v_{vv} = \frac{u}{1 + u^2 + v^2}$.

Sure one may calculate this connection as Levi-Civita connection of the induced Riemannian metric using explicit Levi-Civita formula or using method of Lagrangian of free particle.