Homework 2. Solutions

1 *a) Write down explicit formulae expressing stereographic coordinates for n-dimensional sphere $(x^1)^2 + \ldots + (x^{n+1})^2 = R^2$ of radius R via coordinates $x^1, \ldots, x^{n+1}$ and vice versa.*
(For simplicity you may consider cases $n = 2, 3$.)

b)† *Check that for unit sphere S^2 ($x^2 + y^2 + z^2 = 1$) all the points with rational cartesian coordinates x, y, z have rational stereographic coordinates u, v and vice versa.*

a) Write down the stereographic projection from the North pole of the sphere–point $N = (0, 0, \ldots, R)$ on the plane $x^{n+1} = 0$. Consider the segment ND which intersects the sphere at the point $(x^1, \ldots, x^{n+1})$ ($(x^1)^2 + (x^2)^2 + \ldots + (x^{n+1})^2 = R^2$). This segment intersects the plane $x^{n+1} = 0$ at the point D with coordinates $x^i u^i$ for $i = 1, \ldots, n$. Then comparing similar triangles we have

$$\frac{R}{R - x^{n+1}} = \frac{u^i}{x^i}, \quad \text{i.e. } u^i = \frac{Rx^i}{R - x^{n+1}} \ (i = 1, \ldots, n)$$

and

$$x^i = \frac{u^i(R - x^{n+1})}{R}, \ (i = 1, \ldots, n)\,.$$

Using the fact that $(x^1)^2 + \ldots + (x^{n+1})^2 = R^2$ we come to

$$(x^1)^2 + \ldots + (x^n)^2 = \frac{\left(R - \left(x^{n+1}\right)^2\right)}{R^2} \sum_{i=1}^n \left(u^i\right)^2 = (R - x^{n+1})(R + x^{n+1})\,.$$

Dividing by $R - x^{n+1}$ ($x^{n+1} \neq R$ since North pole is removed) we come to

$$x^{n+1} = \frac{\sum_{i=1}^n (u^i)^2 - R^2}{\sum_{i=1}^n (u^i)^2 + R^2} R\,, \quad x^i = \frac{2u^i R^2}{\sum_{i=1}^n (u^i)^2 + R^2} \ (i = 1, 2, \ldots)$$

For projection with centre in South pole we have to change $x^{n+1} \mapsto -x^{n+1}$.

Write down these formulae for cases $n = 1, 2, 3$,

Case $n = 1$: Circle $x^2 + y^2 = R^2$. Stereographic coordinate t. Centre of projection $(0, R)$:

$$t = \frac{Rx}{R - y}, \qquad \begin{cases} x = \frac{2tR^2}{R^2 + t^2} \\ y = \frac{t^2 - R^2}{t^2 + R^2} R \end{cases} \tag{1}$$

Case $n = 2$: Sphere $x^2 + y^2 + z^2 = R^2$. Stereographic coordinates u, v. Centre of projection $(0, 0, R)$:

$$\begin{cases} u = \frac{Rx}{R - z} \\ v = \frac{Ry}{R - z} \end{cases}, \qquad \begin{cases} x = \frac{2uR^2}{R^2 + u^2 + v^2} \\ y = \frac{2vR^2}{R^2 + u^2 + v^2} \\ z = \frac{u^2 + v^2 - R^2}{u^2 + v^2 + R^2} R \end{cases} \tag{2}$$

Case $n = 3$: 3-dimensional sphere $x^2 + y^2 + z^2 + t^2 = R^2$. Stereographic coordinates u, v, w. Centre of projection $(0, 0, 0, R^2)$:

$$\begin{cases} u = \frac{Rx}{R - t} \\ v = \frac{Ry}{R - t} \\ w = \frac{Rz}{R - t} \end{cases}, \qquad \begin{cases} x = \frac{2uR^2}{R^2 + u^2 + v^2 + w^2} \\ y = \frac{2vR^2}{R^2 + u^2 + v^2 + w^2} \\ z = \frac{2wR^2}{R^2 + u^2 + v^2 + w^2} \\ z = \frac{u^2 + v^2 + w^2 - R^2}{u^2 + v^2 + w^2 + R^2} R \end{cases} \tag{2}$$

b)† We see that from explicit formulae. This is rational transformation of conic surfaces.

2 *Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.*

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

Riemannian metric of Euclidean space is $G = dx^2 + dy^2$.

a) using the angle: In this case parametric equation of circle is $\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}$. Then

$$G = (dx^2 + dy^2)\big|_{x=R\cos\varphi, y=R\sin\varphi} = (d\cos\varphi)^2 + (d\sin\varphi)^2 = R^2 d\varphi^2 \,.$$

b) In stereographic coordinate using (1) we have:

$$G = (dx^2 + dy^2)\big|_{x=x(t), y=y(t)} = \left(d\left(\frac{2tR^2}{R^2+t^2}\right)\right)^2 + \left(d\left(\frac{t^2-R^2}{R^2+t^2}R\right)\right)^2 =$$

$$\left(\frac{2R^2dt}{R^2+t^2} - \frac{4t^2R^2dt}{(R^2+t^2)^2}\right)^2 + \left(\frac{2tRdt}{R^2+t^2} - \frac{2t(t^2-R^2)Rdt}{(R^2+t^2)^2}\right)^2 = \left(\frac{2Rdt}{R^2+t^2}\right)^2 \left[\left(R - \frac{2t^2R}{(R^2+t^2)}\right)^2 + \left(t - \frac{t(t^2-R^2)}{(R^2+t^2)}\right)^2\right]$$

$$= \left(\frac{2Rdt}{R^2+t^2}\right)^2 \left(\frac{R^2(R^2-t^2)^2}{(R^2+t^2)^2} + \frac{4R^4t^2}{(R^2+t^2)^2}\right) = \frac{4R^4dt^2}{(R^2+t^2)^2} \blacksquare$$

3 *Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.*

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

Solution

Riemannian metric of Euclidean space is $G = dx^2 + dy^2 + dz^2$.

a) using the spherical coordinates: In this case parametric equation of sphere is $\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$.

Then

$$G = (dx^2+dy^2+dz^2)\big|_{x=R\sin\theta\cos\varphi, y=R\sin\theta\sin\varphi, z=R\cos\theta} = R^2\left((d\sin\theta\cos\varphi)\right)^2 + R^2\left((d\sin\theta\sin\varphi)\right)^2 + R^2\left((d\cos\theta)\right)^2 =$$

$$R^2\left(\cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi\right)^2 + R^2\left(\cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi\right)^2 + R^2\left(-\sin\theta d\theta\right)^2 = d\theta^2 + \sin^2\theta d\varphi^2 \,.$$

b) in stereographic coordinates using (2) we have $G = (dx^2 + dy^2 + dz^2)\big|_{x=x(u,v), y=y(u,v), z=z(u,v)} =$

$$\left(d\left(\frac{2uR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(\frac{2vR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(\frac{u^2+v^2-R^2}{R^2+u^2+v^2}R\right)\right)^2 =$$

$$R^4\left(\frac{2du}{R^2+u^2+v^2} - \frac{2u(2udu+2vdv)}{(R^2+u^2+v^2)^2}\right)^2 + R^2\left(\frac{2dv}{R^2+u^2+v^2} - \frac{2v(2udu+2vdv)}{(R^2+u^2+v^2)^2}\right)^2 +$$

$$+R^2\left(\frac{2udu+2vdv}{R^2+u^2+v^2} - \frac{(u^2+v^2-R^2)(2udu+2vdv)}{(R^2+u^2+v^2)^2}\right)^2 =$$

$$\frac{4R^4}{(R^2+u^2+v^2)^4}\left\{\left[(R^2-u^2+v^2)du - 2uvdv\right]^2 + \left[(R^2-v^2+u^2)dv - 2uvdu\right]^2 + 4R^2(udu+vdv)^2\right\} =$$

$$\frac{4R^4}{(R^2+u^2+v^2)^4}\left\{\left(R^2+u^2+v^2\right)^2(du^2+dv^2)\right\} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2} \blacksquare$$

Remark

In the case of n-dimensional sphere S^n of radius R in $(n+1)$-dimensional Euclidean space $\mathbf{E}^{n+1}$ (it can be defined by the equation $(x^1)^2 + \ldots + (x^{n+1})^2 = 1$ in cartesian coordinates $x^1, \ldots, x^n, x^{n+1}$) Riemannian metric on this sphere induced by the Euclidean metric in the ambient space in stereographic coordinates has following appearance:

$$G = \left((dx^1)^2 + \ldots + (dx^{n+1})^2\right)\big|_{x^\mu = x^i(u^i)} = \left(\sum_{j=1}^{n}\left(d\left(\frac{2R^2 u^j}{R^2 + \sum_{i=1}^{n}(u^i)^2}\right)\right)\right)^2 + \left(d\left(R\frac{\sum_{i=1}^{n}(u^i)^2 - R^2}{R^2 + \sum_{i=1}^{n}(u^i)^2}\right)\right)^2 =$$

$$= \frac{4R^4 \sum_{i=1}^{n}(du^i)^2}{(R^2 + \sum_{i=1}^{n}(u^i)^2)^2}$$

4 *Consider the surface L which is the upper sheet of two-sheeted hyperboloid in $\mathbf{R}^3$:*

$$L: \qquad z^2 - x^2 - y^2 = 1, \quad z > 0\,.$$

a) Find parametric equation of the surface L using hyperbolic functions $\cosh, \sinh$ *following an analogy with spherical coordinates on the sphere.*

(The surface L sometimes is called pseudo-sphere.)

b) Consider the stereographic projection of the surface L on the plane OXY, i.e. the central projection on the plane $z = 0$ with the centre at the point $(0, 0, -1)$.

Show that the image of projection of the surface L is the open disc $x^2 + y^2 < 1$ in the plane OXY.

a) Parametric equation is $\begin{cases} x = \sinh\theta\cos\varphi \\ y = \sinh\theta\sin\varphi \\ z = \cosh\theta\theta\cos\varphi \end{cases}$ We see that the condition $z^2 - x^2 - y^2 = 1$ is fulfilled. (Compare with equation of sphere in spheric coordinates.)

b) Calculations are very similar to the case of stereographic coordinates for 2-sphere $x^2 + y^2 + z^2 = 1$ of the radius $R = 1$. Stereographic coordinates u, v. Centre of projection $(0, 0, -1)$: We have $\frac{u}{x} = \frac{y}{v} = \frac{1}{1+z}$. Hence $\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}$. Since $x = u(1+z), y = v(1+z)$ then $z^2 - 1 = x^2 + y^2$ and $z^2 - 1 = (u^2 + v^2)(1+z)^2$, i.e. $z = \frac{1+u^2+v^2}{1-u^2-v^2}$. We come to

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}, \qquad \begin{cases} x = \frac{2u}{1-u^2-v^2} \\ y = \frac{2v}{1-u^2-v^2} \\ z = \frac{u^2+v^2+1}{1-u^2-v^2} \end{cases}, \quad |u| < 1, |v| < 1\,. \tag{4}$$

The image of upper-sheet is an open disc $u^2 + v^2 = 1$ since $u^2 + v^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{z^2-1}{(1+z)^2} = \frac{z-1}{z+1}$. Since for upper sheet $z > 1$ then $0 \leq \frac{z-1}{z+1} < 1$.

5 *Consider the pseudo-Riemannian, pseudo-Euclidean metric on $\mathbf{R}^3$ given by the formula*

$$ds^2 = dx^2 + dy^2 - dz^2\,.$$

Calculate the induced metric on the surface L considered in the Exercise 4, and show that it is a Riemannian metric (it is positive-definite).

Perform calculations in spherical-like coordinates (see Exercise 4a) above) and in stereographic coordinates (see exercise 4b) above)

Remark The surface L sometimes is called pseudosphere. The Riemannian metric on this surface sometimes is called Lobachevsky (hyperbolic) metric.

The surface L with this metric realises Lobachevsky (hyperbolic) geometry, where Euclid's 5-th Axiom fails. This Riemannian manifold (manifold+Riemannian metric) we call Lobachevsky (hyperbolic) plane.

In stereographic coordinates we come to realisation of Lobachevsky plane on the disc in $\mathbf{E}^2$. It is so called Poincare model of Lobachevsky geometry.

Solution. The calculations will be very similar to the calculations performed in the exercise 3 above. Just we need consider $\cosh\theta, \sinh\theta$ instead $\cos\theta, \sin\theta$ and and sometimes changes the signs.

First of all consider spherical-like coordinates:

Equation of two-sheeted hyperboloid is $\begin{cases} x = \sinh\theta\cos\varphi \\ y = \sinh\theta\sin\varphi \\ z = \cosh\theta \end{cases}$. Then

$$G = (dx^2+dy^2-dz^2)\big|_{x=\sinh\theta\cos\varphi, y=\sinh\theta\sin\varphi, z=\cosh\theta} = ((d\sinh\theta\cos\varphi))^2+((d\sinh\theta\sin\varphi))^2-((d\cosh\theta))^2 =$$

$$(\cosh\theta\cos\varphi d\theta - \sinh\theta\sin\varphi d\varphi)^2 + (\cosh\theta\sin\varphi d\theta + \sinh\theta\cos\varphi d\varphi)^2 + (\sinh\theta d\theta)^2 = d\theta^2 + \sinh^2\theta d\varphi^2 .$$

matrix of Riemannian metric is $G = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix}$. In the same way as for sphere these coordinates are well-defined in all points except $z = \pm 1$, where $\sin^2\theta = 0$.

Now express Riemannian metric in stereographic coordinates (4):

$$G = (dx^2+dy^2-dz^2)\big|_{x=x(u,v),y=y(u,v),z=z(u,v)} = \left(d\left(\frac{2u}{1-u^2-v^2}\right)\right)^2+\left(d\left(\frac{2v}{1-u^2-v^2}\right)\right)^2-\left(d\left(\frac{u^2+v^2+1}{1-u^2-v^2}\right)\right)^2 =$$

(Compare with calculations for sphere $x^2+y^2+z^2=1$). We have $G = dx^2+dy^2-dz^2 =$

$$\left(\frac{2du}{1-u^2-v^2} + \frac{2u(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 + \left(\frac{2dv}{1-u^2-v^2} + \frac{2v(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 -$$

$$-\left(\frac{2udu+2vdv}{1-u^2-v^2} + \frac{(u^2+v^2+1)(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 = \frac{4(du)^2+4(dv)^2}{(1-u^2-v^2)^2} .$$

(To perform these calculations it is convenient to denote by $s = 1-u^2-v^2$.)

Resume: We come to the induced Riemannian metric on the surface from the pseudo-Riemannian metric in ambient space.

6* *Lobachevsky plane (hyperbolic plane) L in stereographic coordinates can be considered as an open disc $u^2+v^2<1$ in the plane. In the previous exercise in particularly we calculated Riemannian metric of L in these coordinates.*

Find new coordinates x, y such that in these coordinates Lobachevsky plane (hyperbolic plane) can be considered as an upper half plane $y > 0\}$ and write down explicitly Riemannian metric in these coordinates.

Hint: You may use complex coordinates:

$$z = x+iy, \bar z = x-iy, w = u+iv, \bar w = u-iv$$

and find an holomorphic transformation $w = w(z)$ of the open disc $w\bar w < 1$ onto the upper plane $\mathbf{Im}z > 0$.

Solution.

Recall that in the previous exercise we calculated expression for Lobachevsky metric in stereographic coordinates $u, v, u^2+v^2<1$. We come to the answer: $G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}$ (see the previous exercise). (It was realisation of Lobachevsky plane on the Euclidean disc. Sometimes it called Poincare model of Lobachevsky (hyperbolic) geometry.)

In complex coordinates $w = u+iv, \bar{w} = u-iv$ the metric $G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}$ obtained in the exercise 8 can be rewritten $G = \frac{4dwd\bar{w}^2}{(1-w\bar{w})}$. Indeed

$$G = \frac{4dwd\bar{w}}{(1-w\bar{w})^2} = G = \frac{4d(u+iv)d(u-iv)}{(1-(u+iv)(u-iv))^2} = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}\,.$$

It is a beautiful problem in complex analysis: find Mobius transformation $w = \frac{az+b}{cz+d}$ transformation which transforms the interior of circle $w\bar{w} = 1$ into upper half plane $Imz > 0$. One can see that

$$w = \frac{1+iz}{1-iz}, \qquad z = i\frac{1-w}{1+w}$$

is the transformation which we need (Can you find all Mobius transformations which transform upper half plane to the interior of unit circle?.)

Now calculate G in coordinates $z, \bar{z}$. i.e. in coordinates (x, y):

$$G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2} = \frac{4dwd\bar{w}}{(1-w\bar{w})^2}$$

We have

$$dw = d\left(\frac{1+iz}{1-iz}\right) = \frac{2idz}{(1-iz)^2}, d\bar{w} = \frac{-2id\bar{z}}{(1+i\bar{z})^2},$$

$$1-w\bar{w} = 1 - \frac{1+iz}{1-iz}\frac{1-i\bar{z}}{1+i\bar{z}} = \frac{2i(\bar{z}-z)}{(1-iz)(1+i\bar{z})}$$

Hence

$$G = \frac{4dwd\bar{w}}{(1-w\bar{w})^2} = \frac{4\left(\frac{2idz}{(1-iz)^2}\right)\left(\frac{-2id\bar{z}}{(1+i\bar{z})^2}\right)}{\frac{-4(\bar{z}-z)^2}{(1-iz)^2(1+i\bar{z})^2}} = \frac{-4dd\bar{z}}{(\bar{z}-z)^2} = \frac{dx^2+dy^2}{y^2}\,,$$

since $z = x+iy$ and $\bar{z}-z = -2iy$.

We come to the very useful interpretation of hyperbolic geometry: upper half plane in $\mathbf{E}^2$ with metric $G = \frac{dx^2+dy^2}{y^2}$. Later by default we will call "Lobachevsky (hyperbolic) plane" the realisation of Lobachevsky plane as an half-upper plane in $\mathbf{E}^2$ with these coordinates x, y $(y > 0)$ with metric $G = \frac{dx^2+dy^2}{y^2}$.

7 * *Consider the metric induced on one-sheeted hyperboloid $x^2+y^2-z^2 = 1$ embedded in $\mathbf{R}^3$ with the pseudo-Euclidean metric $dx^2+dy^2-dz^2$ (see the exercise 5). Show that this metric is not Riemannian one.*

Solution. One can perform straightforward calculations in spherical-like coordinates: Equation of one-sheeted hyperboloid is $\begin{cases} x = \cosh\theta\cos\varphi \\ y = \cosh\theta\sin\varphi \\ z = \sinh\theta \end{cases}$. Then

$$G = (dx^2+dy^2-dz^2)\big|_{x=\cosh\theta\cos\varphi, y=\cosh\theta\sin\varphi, z=\sinh\theta} = ((d\cosh\theta\cos\varphi))^2+((d\cosh\theta\sin\varphi))^2-((d\sinh\theta))^2 =$$

$$(\sinh\theta\cos\varphi d\theta-\cosh\theta\sin\varphi d\varphi)^2+(\sinh\theta\sin\varphi d\theta+\cosh\theta\cos\varphi d\varphi)^2-(\cosh\theta d\theta)^2 = -d\theta^2+\cosh^2\theta d\varphi^2\,.$$

matrix is $G = \begin{pmatrix} -1 & 0 \\ 0 & \cosh^2\theta \end{pmatrix}$. The condition of positive-definiteness is not fulfilled. This is not Riemannian metric.

Another solution Consider the vectors $\mathbf{e} = \frac{\partial}{\partial y}$ and $\mathbf{f} = \frac{\partial}{\partial z}$ attached at the point $(1, 0, 0)$. One can see that these vectors are tangent to the hyperboloid, but they have the "length" of different sign. (One of these vectors is space-like vector, another time like vector.) We have pseudoriemannian metric at the tangent space spanned by these two vectors.