Elements of Differential Geometry

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Contents

1 Curves in $\mathbf{E^n}$

1.1 Preliminary notes. n-dimensional Euclidean space.

$\mathbf{R}^n$ is a real vector space of n-tuples of real numbers.

It is convenient to distinguish $\mathbf{R}^n$ from the space $\mathbf{E}^n$—the point space of n-tuples. Two points $\mathbf{a}, \mathbf{b} \in \mathbf{E}^n$ define a vector in $\mathbf{R}^n$: if $\mathbf{a} = (a^1, \ldots, a^n)$, $\mathbf{b} = (b^1, \ldots, b^n)$, then the vector $\mathbf{ab}$ attached to the point $\mathbf{a}$ has coordinates $= (b^1 - a^1, b^2 - a^2 \ldots, b^n - a^n)$.

If we fix frame of reference in $\mathbf{E}^n$ then every vector defines a point.

$\mathbf{E^n}$ is Euclidean space: the distance between two points is Pythagorean:

$$||\mathbf{b} - \mathbf{a}|| = \sqrt{(b^1 - a^1)^2 + (b^2 - a^2)^2 + \cdots + (b^n - a^n)^2} \qquad (1)$$

Coordinates $(x^1, \ldots, x^n)$ in $\mathbf{E}^n$ are called cartesian if the distance in these coordinates is expressed by the formula (257).

Two different cartesian coordinates are related with each other by translation and orthogonal transformation (rotation+reflection). E.g. if (x, y) are cartesian coordinates then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a + x\cos\varphi + y\sin\varphi \\ b - x\sin\varphi + y\cos\varphi \end{pmatrix} \qquad (2)$$

are cartesian coordinates too. Usually (by default) we will use cartesian coordinates in $\mathbf{E}^n$.

We recall also very important formula for scalar (inner) product: Let $\mathbf{x}$, $\mathbf{y}$ be two vectors in $\mathbf{E}^n$ with cartesian cooridinates $\mathbf{x} = (x^1, \ldots, x^n)$, $\mathbf{y} = (y^1, \ldots, y^n)$. Let φ be an angle between these vectors. Then scalar product of these vectors is equal to

$$(\mathbf{x}, \mathbf{y}) = |\mathbf{x}||\mathbf{y}|\cos\varphi = x^1y^1 + \ldots x^ny^n \qquad (3)$$

In particular

$$|\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})} \qquad (4)$$

1.2 Curves in $\mathbf{E}^n$

A curve in $\mathbf{E}^n$ with parameter $t \in (a, b)$ is a continuous map

$$\gamma: \qquad (a, b) \to \mathbf{E}^n \qquad \mathbf{r}(t) = (x^1(t), \ldots, x^n(t)), \quad a < t < b \qquad (5)$$

For example consider in $\mathbf{E}^2$ the curve

$$\gamma\colon \qquad (0,2\pi)\to\mathbf{E}^2 \qquad \mathbf{r}(t)=(R\cos wt, R\sin wt),\ 0\le t<2\pi \tag{6}$$

The image of this curve is the circle of the radius R. It can be defined by the equation:

$$x^2+y^2=R^2 \tag{7}$$

To distinguish between curve and its image we say that curve γ in (5) is *parameterised* curve or *path*. We will call the image of the curve *unparameterised curve* or just curve (see for details the next subsection). It is very useful to think about parameter t as a "time" and consider parameterised curve like *point moving along a curve.* Unparameterised curve is the trajectory of the moving point.

We consider only smooth curves, i.e. curves $\mathbf{r}(t)=(x^1(t),\dots,x^n(t))$ such that all functions $x^i(t)$, $(i=1,2,\dots,n)$ are smooth functions. (Function is called smooth if it has derivatives of arbitrary order.)

Velocity and acceleration

Let $\gamma\colon \quad \mathbf{r}=\mathbf{r}(t)$ be a curve in $\mathbf{E}^n$.

Velocity $\mathbf{v}(t)$ it is the vector

$$\mathbf{v}(t)=\frac{d\mathbf{r}}{dt}=\big(\dot{x}^1(t),\dots,\dots\dot{x}^n(t)\big)=\big(v^1(t),\dots,v^n(t)\big) \tag{8}$$

in $\mathbf{E}^n$. Velocity vector is *tangent vector to the curve.*

We consider also *acceleration vector*:

$$\mathbf{a}(t)=\frac{d^2\mathbf{r}}{dt^2}=\left(\frac{d^2x^1(t)}{dt^2},\dots,\frac{d^2x^n(t)}{dt^2}\right) \tag{9}$$

The value of velocity vector is a *speed*:

$$|\mathbf{v}|=\sqrt{(v^1)^2+\dots+(v^n)^2}\,.$$

Example Consider the following curve in $\mathbf{E}^2$

$$\mathbf{r}(t)\colon \qquad \begin{cases} x(t)=R\cos wt\\ y(t)=R\sin wt\end{cases} \tag{10}$$

One can see that for this curve

$$\mathbf{v}(t) = \begin{pmatrix} -wR\sin wt \\ wR\cos wt \end{pmatrix}, \quad \mathbf{a}(t) = \begin{pmatrix} -w^2R\cos wt \\ -w^2R\sin wt \end{pmatrix} = -w^2\mathbf{r}(t) \tag{11}$$

We come to the formula well-known from the school: If particle moves with constant speed along circle then its acceleration (centripetal acceleration) is orthogonal to the tangent vector.

The velocity vector changes only its value under changing of parameterisation, i.e. it is multiplied on a scalar coefficient. In general not only the value of acceleration but its direction changes if we change a parameterisation of the curve (See the next two subsection)

1.3 Reparameterisation

One can move along trajectory with different velocities, i.e. one can consider different parameterisation. E.g. consider

$$\gamma_1\colon \qquad \begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} ,0 < t < 1 \qquad \gamma_2\colon \qquad \begin{cases} x(t) = \sin t \\ y(t) = \sin^2 t \end{cases} ,0 < t < \frac{\pi}{2} \tag{12}$$

Images of these two parameterised curves are the same. In both cases point moves along a piece of the same parabola but with different velocities.

Definition

Two smooth curves

$\gamma_1\colon \quad \mathbf{r}_1(t)\colon (a_1, b_1) \to \mathbf{E}^n$ and

$\gamma_2\colon \quad \mathbf{r}_2(t)\colon (a_2, b_2) \to \mathbf{E}^n$ are called equivalent if there exists reparameterisation map:

$$\varphi\colon (a_2, b_2) \to (a_1, b_1),$$

such that

$$\mathbf{r}_2 = \mathbf{r}_1 \circ \varphi, \qquad r_2(t) = r_1(\varphi(t)) \tag{13}$$

Reparameterisation φ is diffeomorphism, i.e. φ has derivatives of all orders and first derivative $\varphi'(t)$ is not equal to zero.

E.g. curves in (12) are equivalent because a map $\varphi(t) = \sin t$ transforms first curve to the second.

Equivalence class of equivalent parameterised curves is called non-parameterised curve.

Non-formally: Two curves are equivalent curves (belong to the same equivalence class) if these parameterised curves (paths) have the same images. We come to equivalent curves if we consider the movement along the same trajectory with *different speeds.*

Non-parameterised curve—it is trajectory of point.

Or in other words: *two equivalent curves have the same image. They define the same set of points in* $\mathbf{E}^n$*. Different parameters correspond to moving along curve with different velocity.*

Example

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}, 0 < \theta < \pi, \quad \begin{cases} x = u \\ y = \sqrt{1-u^2} \end{cases}, -1 < u < 1, \tag{14}$$

$$\begin{cases} x = \tan t \\ y = \sqrt{1-\tan^2 t} \end{cases}, -\frac{\pi}{4} < t < \frac{\pi}{4} \tag{15}$$

These three parameterised curves,(paths) define the same non-parameterised curve: the upper piece of the circle: $x^2+y^2 = 1, y > 0$. In the first case point moves with constant speed $|\mathbf{v}(\theta)| = 1$ and acceleration is orthogonal to the velocity and it is directed to the centre.

In the second and third case speed is not constant. Hence acceleration is not orthogonal to the velocity. It has tangential component also.

Very practical observation:

- if the angle between velocity and acceleration vector is right (i.e. the scalar product $(\mathbf{v}, \mathbf{a})$ of acceleration and velocity vectors is equal to zero) then the speed is constant: velocity vector only may change its direction.
- if the angle between velocity and acceleration vector is acute (i.e. the scalar product $(\mathbf{v}, \mathbf{a})$ of acceleration and velocity vectors is a positive number) then the speed is increasing
- if the angle between velocity and acceleration vector is obtuse (i.e. the scalar product $(\mathbf{v}, \mathbf{a})$ of acceleration and velocity vectors is a negative number) then the speed is decreasing

(To see it: it is very useful to make exercise about moving of point along ellipse: $x = a\cos t, y = b\sin t$ (see Homework 2),)

1.4 Tangent line and Osculating plane

What happens with velocity and acceleration if we change parameterisation?

Velocity vector changes its value but does not change direction, in a sense that new and former vectors are proportional (are on the one line) Acceleration vector in general changes its direction. (E.g. if particle moves along circle increasing its speed then acceleration vector possesses not only orthogonal centripetal component but also tangential component)

Consider in $\mathbf{E}^n$ arbitrary smooth path (parameterised curve) $\mathbf{r}(t)$, $a < t < b$. Change parameterisation: Suppose $t = t(\tau)$, $a' < \tau < b'$. Consider paths $\mathbf{r}_1(t) = \mathbf{r}(t)$ and $\mathbf{r}_2(\tau) = \mathbf{r}(t(\tau))$. It is different parameterisation of the same non-parameterised curve.

Compare velocity and acceleration vectors $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$, $\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$ with velocity and acceleration vectors $\mathbf{v}(\tau) = \frac{d\mathbf{r}_2(\tau)}{d\tau}$, $\mathbf{a}(\tau) = \frac{d^2\mathbf{r}_2(\tau)}{d\tau^2}$ at the same point $\mathbf{r}(t) = \mathbf{r}(t(\tau))$.

Simple calculations using chain rule show that

$$\mathbf{v}(\tau) = \frac{d\mathbf{r}_2(\tau)}{d\tau} = \frac{d\mathbf{r}(t(\tau))}{d\tau} = \frac{dt(\tau)}{d\tau} \cdot \frac{d\mathbf{r}(t)}{dt}\Big|_{t=t(\tau)}$$

Hence

$$\mathbf{v}(\tau)\Big|_{\tau} = t_\tau(\tau)\mathbf{v}(t)\Big|_{t=t(\tau)} \tag{16}$$

For the case of acceleration calculations are simple too. Just little bit more boring:

$$\mathbf{a}(\tau) = \frac{d^2\mathbf{r}_2(\tau)}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dt(\tau)}{d\tau} \cdot \frac{d\mathbf{r}(t)}{dt}\Big|_{t=t(\tau)}\right) =$$

$$\left(\frac{dt(\tau)}{d\tau}\right)^2 \cdot \frac{d^2\mathbf{r}(t)}{dt^2}\Big|_{t=t(\tau)} + \frac{d^2t(\tau)}{d\tau^2} \cdot \frac{d\mathbf{r}_1(t)}{dt}\Big|_{t=t(\tau)}$$

Hence

$$\mathbf{a}(\tau)\Big|_{\tau} = t_{\tau\tau}(\tau)\mathbf{v}(t)\Big|_{t=\varphi(\tau)} + t_\tau^2(\tau)\mathbf{a}(t)\Big|_{t=\varphi(\tau)} \tag{17}$$

Combine together the formulae (16) and (17):

$$\begin{pmatrix} \mathbf{v}'(\tau) \\ \mathbf{a}'(\tau) \end{pmatrix} = \begin{pmatrix} t_\tau & 0 \\ t_{\tau\tau} & t_\tau^2 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v}(t) \\ \mathbf{a}(t) \end{pmatrix} \Big|_{t=t(\tau)} \tag{18}$$

Now we can summarize formulae (16), (17) and (18) in the

Proposition-Definition

Under changing of parameterisation a velocity vector in a given point of the curve is multiplied on a scalar number $\frac{dt}{d\tau}$. Velocity vector does not change its direction[1]*. It changes its value (speed). Velocity vector defines a line which does not depend on parameterisation. This line is called a* **tangent line** *to the given point of the curve.*

Acceleration vector in a given point of the curve in new parameterisation becomes equal to linear combination of velocity and acceleration vectors in former parameterisation. The plane formed by acceleration and velocity vectors remains unchanged under reparameterisation. This plane is called **osculating** *plane.*

Contrary to velocity vector acceleration vector *changes it direction under changing of parameterisation. But nevertheless under changing of parameterisation it does not quit osculating plane.* [2]

Formulae (16), (17) are very important for studying functionals on curves which *does not depend on parameterisation.* (A typical problem of differential geometry of curves)

Consider explicit formulae for tangent line and osculating plane:

Let $\mathbf{r} = \mathbf{r}(t)$ be a curve in $\mathbf{E}^n$. Let $\mathbf{r}_0 = \mathbf{r}(t_0)$ be a point on this curve.

- *tangent line*: line spanned by velocity vector. If $\mathbf{v}_0$ is velocity vector at the given point $\mathbf{r}_0 = \mathbf{r}(t_0)$ then this line is defined by the equation:

$$\mathbf{l}(t) = \mathbf{r}_0 + \mathbf{v}_0(t - t_0)\,, \qquad \mathbf{v}_0 = \frac{d\mathbf{r}(t)}{dt}\Big|_{t=t_0} \tag{19}$$

[1]We understand direction in a wide sense: two vectors **a** and **b** have the same direction if they are proportional. If $\mathbf{a} = \mu\mathbf{b}$ with $\mu < 0$ (this corresponds to the case where we consider reparameterisation with $t_\tau < 0$) then in narrow sense α and **b** have opposite directions

[2]We do not consider degenerate case where acceleration vector belongs to tangent line and osculating plane is not defined

- *osculating plane*: plane spanned by acceleration and velocity vectors. If $\mathbf{v}_0$ is velocity vector at the given point $\mathbf{r}_0 = \mathbf{r}(t_0)$, and $\mathbf{a}_0$ is acceleration vector at this point then this plane is defined by the equation:

$$\mathbf{L}(\xi, \eta) = \mathbf{r}_0 + \mathbf{v}_0\xi + \mathbf{a}_0\eta, \qquad (\xi, \eta \in \mathbf{R}) \tag{20}$$

Geometrical meaning of tangent line and osculating plane.

Tangent line has first order contact (touching) with a curve at the touching point:

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}(t - t_0) + o(t) \tag{21}$$

Now we are ready for giving another definition of tangent line:

Consider two points on the curve the fixed point $\mathbf{r}(t_0)$ *and second point* $\mathbf{r}(t_0 + \delta t)$ *Consider lines which pass through this points. Tangent line at the point* $\mathbf{r}(t_0)$ *of the curve* $\mathbf{r}(t)$ *is a limit of these lines when* $\delta t \to 0$.

This definition intuitively is more understandable in spite of the fact that it is not effective for working out formulae.

Now two words about geometrical meaning of osculating plane.

One can see that oscullating plane has second order contact (touching) with a curve at the touching point. Indeed Let $\mathbf{r}(t)$ be an arbitrary point of the curve in the vicinity of the given point $\mathbf{r}_0$. Then the distance between the point $\mathbf{r}(t)$ and the oscullating plane $\mathbf{L}(\xi, \eta)$ defined by (20) is less or equal than distance between points $\mathbf{r}(t)$ and the point on the plane with coordinates $\xi = (t - t_0), \eta = \frac{1}{2}(t - t_0)^2$. Hence

$$d(\mathbf{r}(t), \mathbf{L}) \leq |\mathbf{r}(t) - \mathbf{r}(t_0) - \mathbf{v}(t - t_0) - \frac{1}{2}(t - t_0)^2| = o((t - t_0)^2)$$

Remark Osculating plane is not defined at the points where acceleration and velocity vectors are proportional each other, i.e. points where curvature of the curve is equal to zero.

The following exercise is very instructive:

Exercise Find coefficients A, B, C, D such that plane $Ax + By + Cz = D$ is an osculating plane to the curve $x = f(t), y = g(t), z = z(t)$

(See solution in solutions of Homework 2.)

1.5 Length of the curve

Velocity vector measures the length of the curve. One can consider length δl of the small arc of the curve as $\delta l \approx |v|\delta t$.

This leads to the definition of the length of the curve: If $\mathbf{r}(t), a \leq t \leq b$ is a parameterisation of the curve L and $\mathbf{v}(t)$ velocity vector then length of the curve is equal to the integral of of $|\mathbf{v}(t)|$ over curve:

$$\text{Length of the curve } L = \int_a^b |\mathbf{v}(t)| dt = \tag{22}$$

$$\int_a^b \sqrt{\left(\frac{dx^1(t)}{dt}\right)^2 + \left(\frac{dx^2(t)}{dt}\right)^2 + \cdots + \left(\frac{dx^n(t)}{dt}\right)^2} dt$$

Note that formula above is *reparameterisation* invariant. The length of the image of the curve does not depend on parameterisation. Indeed consider curve $\mathbf{r}_1 = \mathbf{r}_1(t)$, $a_1 \leq t \leq b_1$. Let $t = t(\tau)$, $a_2 < \tau < b_2$ be another parameterisation of the curve $\mathbf{r} = \mathbf{r}(t)$, In other words we have two different parameterised curves $\mathbf{r}_1 = \mathbf{r}_1(t), a_1 \leq t \leq b_1$ and $\mathbf{r}_2 = \mathbf{r}_1(t(\tau))$, $a_2 \leq \tau \leq \mathbf{b}_2$ such that their images coincide (See (13)). Show that length of the curve $\mathbf{r}_2(\tau)$ coincide with the length of the curve $\mathbf{r}_1(t)$. Note that under reparameterisation velocity vector is multiplied on t_τ (see (16)):

$$\mathbf{v}_2 = \frac{d\mathbf{r}_2}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{r}_1}{dt} = t_\tau(\tau)\mathbf{v}_1(t(\tau))$$

Hence

$$L_1 = \int_{a_1}^{b_1} \mathbf{v}_1(t)| dt = \int_{a_2}^{b_2} |\mathbf{v}_1(t)| \frac{dt(\tau)}{d\tau} d\tau = \int_{a_2}^{b_2} |t_\tau \mathbf{v}_1(t)| d\tau = \int_{a_2}^{b_2} |\mathbf{v}_2(\tau)| d\tau = L_2. \tag{23}$$

Remark In the formula above we suppose that $t_\tau > 0$. If it is not the case then $a_2 > b_2$ and we have to put $|t_\tau|$ instead t_τ and change a sign of integral

Consider also simple examples:

1) interval of line in $\mathbf{E}^2$ which connects the points (a_1, b_1) and (a_2, b_2)

$x = a_1 + t, y = b_1 + kt$, where $k = \frac{b_2 - b_1}{a_2 - a_1}$, $0 \leq t \leq a_2 - a_1$. (At $t = 1$, $x = a_1 + (a_2 - a_1) = a_2$, $y = b_1 + k(a_2 - a_1) = b_2$)

The integral above gives that

$$L = \int_{a_1}^{a_2} \sqrt{v_x^2 + v_y^2} dt = \int_{a_1}^{a_2} \sqrt{1 + k^2} dt = (a_2 - a_1)\sqrt{1 + k^2} = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)}$$

It is just the length which is given by Pythagoras formula.

2) Arc of the circle of the radius R with angle φ: $x = R\cos t, y = R\sin t$ $0 \le t \le \varphi$. Calculating integral

$$L = \int_0^\varphi \sqrt{v_x^2 + v_y^2} dt = \int_0^\varphi \sqrt{v_x^2 + v_y^2} dt = \int_0^\varphi \sqrt{R^2 \sin^2 t + R^2 \sin^2 t} dt = R\varphi$$

we come to the formula which already know.

1.6 Natural parameterisation

Non-parameterised curve can be parameterised in many different ways. (See e.g.(14)).

Is there any distinguished parameterisation?

Yes. there is.

Let $\gamma : \mathbf{r}(t) = (x^1(t), \dots, x^n(t)), a < t < b$ be some curve in $\mathbf{E}^n$.

Using (22) consider the following parameter on the curve γ:

$$s(t) =$$

$$\{\text{length of the arc of the curve for parameter less or equal to } t\} = \quad (24)$$

$$= \int_a^t \sqrt{\left(\frac{dx^1(\tau)}{d\tau}\right)^2 + \left(\frac{dx^2(\tau)}{d\tau}\right)^2 + \dots + \left(\frac{dx^n(\tau)}{d\tau}\right)^2} d\tau = \int_a^t |\mathbf{v}(\tau)| d\tau \,. \quad (25)$$

Examples Consider circle: $x = R\cos t, y = R\sin t$ in E^2. Then we come to the obvious answer

$$s(t) = \{\text{length of the arc of the circle for parameter less or equal to } t\} =$$

$$\int_0^t \sqrt{\left(\frac{dx(\tau)}{d\tau}\right)^2 + \left(\frac{dy(\tau)}{d\tau}\right)^2} d\tau = \int_0^t \sqrt{R^2 \sin^2 \tau + R^2 \cos^2 \tau} d\tau = \int_a^t R d\tau = Rt$$

Another example:

Consider arc of the parabola $x = t, y = t^2, 0 < t < 1$:

$$s(t) = \{\text{length of the arc of the curve for parameter less or equal to } t\} = \quad (26)$$

$$\int_0^t \sqrt{\left(\frac{dx(\tau)}{d\tau}\right)^2 + \left(\frac{dy(\tau)}{d\tau}\right)^2} d\tau =$$

$$\int_0^t \sqrt{1+4\tau^2}d\tau = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{4}\log\left(2t+\sqrt{1+4t^2}\right)$$

The first example was very simple. The second is harder to calculate [3]. In general case natural parameter is not so easy to calculate. But it is very important for studying properties of curves.

Natural parameterisation is distinguished. Later we will often use the following very important property of natural parameterisation:

Proposition *If a curve is given in natural parameterisation then the speed is equal to* 1 *and acceleration is orthogonal to velocity:*

$$\mathbf{v}(s)\cdot\mathbf{v}(s) \equiv 1, \quad \text{i.e. } |\mathbf{v}(s)| \equiv 1\,, \tag{27}$$

$$\mathbf{v}(s)\cdot\mathbf{a}(s) = 0\,, \quad \text{i.e.}\textit{acceleration is orthogonal to velocity:}. \tag{28}$$

The first relation is just definition of natural parameter and speed. The second relation means that value of the speed does not change.

Formal proof: let s be a natural parameter for (non-parameterised) curve $\mathbf{r} = \mathbf{r}(s)$. Then by definition

$$s = \int_a^s |\mathbf{v}(\tau)|d\tau$$

Differentiating by s we see that

$$\frac{ds}{d\tau} = \text{length of velocity vector. It is equal to one:}|\mathbf{v}(s)| = 1 \tag{29}$$

We come to (27). Differentiating the equation (27) by s we come to the condition (28).

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[3]Denote by $I = \int_0^t \sqrt{1+4\tau^2}d\tau$. Then integrating by parts we come to:

$$I = t\sqrt{1+4t^2} - \int \frac{4\tau^2}{\sqrt{1+4\tau^2}}d\tau = t\sqrt{1+4t^2} - I + \int \frac{1}{\sqrt{1+4\tau^2}}d\tau\,.$$

Hence

$$I = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{2}\int \frac{1}{\sqrt{1+4\tau^2}}d\tau\,.$$

and we come to the answer.

Note Very useful formulae which follow immediately from definition of natural parameter: Let t be an arbitrary parameter then it follows from definition (25) that

$$\frac{ds(t)}{dt} = |\mathbf{v}(t)|, \quad \frac{dt(s)}{ds} = \frac{1}{|\mathbf{v}(s)|} \tag{30}$$

1.7 Invariants of curves. Curvature

How to find invariants of non-parameterised curve, i.e. magnitudes which depend on the points of non-parameterised curve but which do not depend on parameterisation?

Answer at the first sight looks very simple: Consider the distinguished natural parameterisation $\mathbf{r} = \mathbf{r}(s)$ of the curve. Then arbitrary functions on $x^i(s)$ and its derivatives do not depend on parameterisation. But the problem is that it is not easy to calculate natural parameter explitly (See e.g. calculations of natural parameter for parabola in the previous subsection). So it is preferable to know how to construct these magnitudes in arbitrary parameterisation, i.e. construct functions $f(\frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \dots)$ such that they *do not depend on parameterisation*. We want to define curvature.

First formulate reasonable conditions on curvature: (31)

- it have to be a function of the points of the curve
- it does not depend on parameterisation (or at most only a sign depends on parameterisation)
- curvature of the line must be equal to zero
- curvature of the circle with radius R must be equal to $1/R$

We first give definition of curvature in natural parameterisation. Then consider it for arbitrary parameterisation.

For a given non-parameterised curve consider natural parameterisation $\mathbf{r} = \mathbf{r}(s)$. We know already that velocity vector has length 1 and acceleration vector is orthogonal to curve in natural parameterisation (see (27) and (28)).

Definition. The curvature of the curve in a given point is equal to the modulus (length) of acceleration vector in natural parameterisation.

Namely, let $\mathbf{r}(s)$ be natural parameterisation of this curve. Then curvature at every point $\mathbf{r}(s)$ of the curve is equal to the length of acceleration vector:

$$k = |\mathbf{a}(s)|, \qquad \mathbf{a}(s) = \frac{d^2\mathbf{r}(s)}{ds^2} \tag{32}$$

First check that it corresponds to our intuition (see reasonable conditions (31)).

It does not depend on parameterisation by definition.

It is evident that for the line in normal parameterisation $x^i(t) = x^i_0 + b^i t$ with $\sum b^i b^i = 1$ the acceleration is equal to zero.

Now check that the formula (32) gives a natural answer for circle: if radius is equal to R then curvature k is equal to $1/R$.
For circle of radius R in natural parameterisation

$$\mathbf{r} = \mathbf{r}(s) = (x(s), y(s)), \quad \text{where} \quad x(s) = R\cos\frac{s}{R}, \quad y(s) = R\sin\frac{s}{R}$$

(length of the arc of the angle θ of the circle is equal to $s = R\theta$.) Then

$$\mathbf{a}(s) = \frac{d\mathbf{r}^2(s)}{ds^2} = \left(-\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right)$$

and for curvature

$$k = |\mathbf{a}(s)| = \frac{1}{R} \tag{33}$$

we come to the answer which agrees with our intuition.

Geometrical meaning of curvature: We will consider this question later in more detail. But even now it is easy to see from this example that $\frac{1}{k}$ is *just a radius of the circle which has second order touching to curve.*

Example: parabola: $y = ax^2.(a > 0)$ Calculate the curvature at the point $(0, 0)$.

The straightforward calculations are very long. Later we will return to this example when we find a formula expressing curvature in arbitrary parameterisation. But try to guess an answer using geometrical meaning of curvature *that* $\frac{1}{k}$ *is just a radius of the circle which has second order touching to curve.*

Consider a circle given by equation

$$x^2 + (y - R)^2 = R^2 \tag{34}$$

It is evident that this circle touches parabola

$$y = ax^2 \tag{35}$$

at the point $(0,0)$. Choose radius R such that this will be a touching of the second order. Opening brackets in (34) we come to quadratic equation $y^2 - 2yR + x^2 = 0$ and expressing y via x we come to $y = R \pm \sqrt{R^2 - x^2}$. The lower part of circle touches to parabola. Hence:

$$y = R - \sqrt{R^2 - x^2} = R - R\left(\sqrt{1 - \frac{x^2}{R^2}}\right) = \tag{36}$$

$$R - R\left(1 - \frac{x^2}{2R^2} + o(x^2)\right) = \frac{x^2}{2R} + o(x^2) \tag{37}$$

Comparing with (35) we see that $a = \frac{1}{2R}$ and $k = \frac{1}{R} = 2a$.

Formula (5) looks simple but it is hard to work with it (because we have to consider a curve in a natural parameterisation).

Try to rewrite formula for curvature in arbitrary parameterisation.

Let $[\gamma]$ be non-parameterised simple regular curve (equivalence class of parameterised curves). Let $\mathbf{r}(t)$ be any parameterisation of this curve. Consider arbitrary point $\mathbf{r}(t_0)$ of this curve.

Straightforward attack. Instead considering explicitly natural parameterisation of the curve we just try to rewrite the formula in definition (32) using chain rule and the relation (30): Using chain rule we calculate according definition (32) the curvature. To avoid confusion we denote here by $\mathbf{v} = \mathbf{v}(t), \mathbf{a} = \mathbf{a}(t)$ velocity and acceleration vectors in a given parameterisation t. We denote by $\mathbf{A}$(s) the acceleration vector in natural parameterisation. First using (30) we calculate the vector $\mathbf{A}(s) = \frac{d^2\mathbf{r}}{ds^2}$ (the vector of acceleration in normal parameterisation). Then we calculate its length (curvature):

$$\mathbf{A}(s) = \frac{d^2\mathbf{r}(t(s))}{ds^2} = \frac{d}{ds}\left(\frac{dt}{ds}\frac{d\mathbf{r}(t)}{dt}\right) = \frac{d}{ds}\left(\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}\right) = \frac{1}{|\mathbf{v}(t)|}\frac{d}{dt}\left(\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}\right) =$$

$$\frac{1}{|\mathbf{v}|}\left(\frac{\mathbf{a}}{|\mathbf{v}|} - \frac{\mathbf{v}(\mathbf{a}\cdot\mathbf{v})}{|\mathbf{v}|^3}\right) = \frac{\mathbf{a}\mathbf{v}^2 - \mathbf{v}(\mathbf{a}\mathbf{v})}{\mathbf{v}^4} \tag{38}$$

$\mathbf{v}^2 = (\mathbf{v}, \mathbf{v}) = |\mathbf{v}|^2, (\mathbf{v}\,\mathbf{a})$ are inner (scalar) products.

Curvature is equal to

$$k(\mathbf{r}(t)) = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \frac{\sqrt{\mathbf{a}^2\mathbf{v}^2 - (\mathbf{a} \cdot \mathbf{v})^2}}{|\mathbf{v}|^3} \tag{39}$$

Try to understand the geometrical meaning of the final answer. Show that the numerator of the fraction in RHS of the last formula is nothing but **area of the parallelogram formed by the vectors $\mathbf{v}, \mathbf{a}$**

Indeed recall the formula from elementary geometry: Area of the parallelogram is equal to the height multiplied on the length of the side, i.e. area of the parallelogram formed by the vectors $\mathbf{v}, \mathbf{a}$ is equal to the length of the vector $\mathbf{v}$ multiplied on the length of the vector $\mathbf{a}$ and multiplied on the sinus of the angle between these vectors. On the other hand inner product of vectors $\mathbf{v}, \mathbf{a}$ is equal to the product of their length on the cosines of the angle between them: $\mathbf{v} \cdot \mathbf{a} = |\mathbf{v}||\mathbf{a}| \cos\varphi$. Hence

$$\text{area of the parallelogram formed by the vectors } \mathbf{v}, \mathbf{a} =$$

$$|\mathbf{v}||\mathbf{a}| \sin\varphi = |\mathbf{v}||\mathbf{a}|\sqrt{1 - \cos^2\varphi} = |\mathbf{v}||\mathbf{a}|\sqrt{1 - \left(\frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}||\mathbf{a}|}\right)^2} = \sqrt{\mathbf{a}^2\mathbf{v}^2 - (\mathbf{a} \cdot \mathbf{v})^2} \tag{40}$$

We come to the

Theorem *Let $[\gamma]$ be non-parameterised curve (equivalence class of parameterised curves). Let $\mathbf{r}(t)$ be any parameterisation of this curve. Then curvature at the point $\mathbf{r}(t_0)$ of the curve is equal to the area of parallelogram formed by the vectors $\mathbf{a}, \mathbf{v}$ at this point divided by the cube of the speed $\mathbf{v}$:*

$$k\Big|_{\mathbf{r}=\mathbf{r}(t_0)} = \frac{\textit{Area of parallelogram formed by the vectors } \mathbf{v} \textit{ and } \mathbf{a}}{\textit{Cube of the speed}} =$$

$$\frac{\sqrt{\mathbf{v}^2\mathbf{a}^2 - (\mathbf{v} \cdot \mathbf{a})^2}}{|\mathbf{v}|^3} \tag{41}$$

where vectors $\alpha, \mathbf{v}$ are considered at the point $\mathbf{r}(t_0)$: $\mathbf{v} = \frac{d\mathbf{r}}{dt}\Big|_{t=t_0}$, $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}\Big|_{t=t_0}$.

Previous calculations were in fact proof of this Theorem. But these calculations were just straightforward calcualtions.

One can give the proof of the Theorem independent on previous calculations which is beautiful and illuminating.

Proof: In natural parameterisation according to (27) acceleration is orthogonal to velocity and speed is equal to 1: $(\mathbf{a}, \mathbf{v}) = 0, |\mathbf{v}| = 1$. Hence RHS of (41) coincides with (32) in natural parameterisation. It remains to prove that it is independent on parameterisation. Consider arbitrary reparameterisation. According to formulae (18) $\mathbf{v} \to t_\tau \mathbf{v}$, $\mathbf{a} \to t_\tau^2 \mathbf{a} + t_\tau \mathbf{v}$. The area of parallelogram will be multiplied on t_τ^3. (Transformation $\mathbf{v} \to \mathbf{v}, \mathbf{a} \to \mathbf{a} + \mu \mathbf{v}$ does not change the area). The denominator of the fraction (41) will be multiplied on the same number t_τ^3. Hence RHS of (18) is independent of parameterisation. ∎

Formula (41) is a workable definition of curvature.

1.8 Curvature of curves in $\mathbf{E}^3$ and $\mathbf{E}^2$. Signed curvature for plane curves (curves in $\mathbf{E}^2$)

If $\mathbf{r} = \mathbf{r}(t)$ is curve on $\mathbf{E}^3$ then one can express area of parallelogram via cross-product. Recall that if $\mathbf{a}, \mathbf{b}$ are two vectors in three-dimensional Euclidean space. then one can consider their cross-product (vector product): vector $\mathbf{a} \times \mathbf{b}$ (sometimes denoted by $[\mathbf{a}, \mathbf{b}]$ which is defined by the following properties:

- its length is equal to the area of parallelogram formed by the vectors $\mathbf{a}, \mathbf{b}$
- it is orthogonal to the plane spanned by the vectors $\mathbf{a}, \mathbf{b}$
- The vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ form an basis such that it has the same orientation that the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

One can see that

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} = (a_y b_z - a_z b_y)\mathbf{e}_x + (a_z b_x - a_x b_z)\mathbf{e}_y + (a_x b_y - a_y b_x)\mathbf{e}_z, \tag{42}$$

Its length is given by the formula

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_x b_y - a_y b_x)^2} = \tag{43}$$

$$\sqrt{\left(a_x^2+a_y^2+a_z^2\right)\left(b_x^2+b_y^2+b_z^2\right)-\left(a_xb_x+a_yb_y+a_zb_z\right)^2}=\sqrt{\mathbf{a}^2\mathbf{b}^2-(\mathbf{ab})} \tag{44}$$

We see that for curve in $\mathbf{E}^3$ curvature (41) can be expressed via cross-product in a very compact way:

$$k=\frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} \tag{45}$$

Now consider a case if curve belongs to the plane, is so called plane curve. WLOG suppose that curve belongs to the plane OXY.

Recall basic formulae about area of parallelogram: *area of parallelogram formed by vectors* $\mathbf{a},\mathbf{b}$ *(*$\mathbf{a},\mathbf{b}\in OXY$ *is equal to the determinant of the matrix*

$$\begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} \tag{46}$$

(Compare this formula with (42) in the case if $a_z=b_z=0$)

One have to distinguish between area of *oriented parallelogram* and non-oriented parallelogram. Area of non-oriented parallelogram is non-negative number. It is equal to module of the determinant (46).

The area of oriented parallelogram is defined by the orientation of pair of vectors $(\mathbf{a},\mathbf{b})$.

The determinant (46) may take positive or negative values. E.g. for the vectors $\mathbf{a}=\mathbf{e}_x,\mathbf{b}=\mathbf{e}_y$ determinant is equal to 1, if we change ordering and consider $\mathbf{a}=\mathbf{e}_y,\mathbf{b}=\mathbf{e}_x$ then we come to -1.

Assume that basic vectors $\mathbf{e}_x,\mathbf{e}_y$ are oriented in standard way, i.e. rotation from $\mathbf{e}_x$ to $\mathbf{e}_y$ is counter clock wise. Then it is easy to see that

Area of oriented parallelogram formed by vectors $\mathbf{a}$*,* $\mathbf{b}$ *is positive if rotation from* $\mathbf{a}$ *to* $\mathbf{b}$ *is counter clock wise. Respectively the area of oriented parallelogram formed by vectors* $\mathbf{a}$*,* $\mathbf{b}$ *is negative if rotation from* $\mathbf{a}$ *to* $\mathbf{b}$ *is clock wise.*

We say that the ordered pair of vectors $\mathbf{a}$, $\mathbf{b}$ on the plane OXY has positive orientation if this ordered pair has the same orientations as the ordered pair of basic vectors $\mathbf{e}_x,\mathbf{e}_y$, i.e. rotation from $\mathbf{a}$ to $\mathbf{b}$ is counter clock wise and determinant (46) is positive. Respectively we say that the ordered pair of vectors $\mathbf{a}$, $\mathbf{b}$ on the plane OXY has negative orientation if this pair has the same orientations as the ordered pair of basic vectors $\mathbf{e}_y,\mathbf{e}_x$, i.e. rotation from $\mathbf{a}$ to $\mathbf{b}$ is clock wise and determinant (46) is negative.

Example The pairs $\{\mathbf{a},\mathbf{b}\}$ and $\{\mathbf{b},\mathbf{a}\}$ have opposite orientations. The pairs

$\{\mathbf{a},\mathbf{b}\}$ and $\{-\mathbf{a},\mathbf{b}\}$ have opposite orientations. The pairs $\{\mathbf{a},\mathbf{b}\}$ and $\{-\mathbf{b},\mathbf{a}\}$ have the same orientations.

Return to curvature.

For curve in OXY $\mathbf{r}(t); x = x(t), y = y(t)$ curvature (length of the acceleration vector in normal parameterisation) according to (41) and (46) is equal to the area of non-oriented parallelogram divided by the cube if the velocity:

$$k = \pm \frac{\det \begin{pmatrix} x_t & y_t \\ x_{tt} & y_{tt} \end{pmatrix}}{|\mathbf{v}(t)|^3} = \frac{|x_t y_{tt} - x_{tt} y_t|}{(x_t^2 + y_t^2)^{3/2}} \tag{47}$$

Signed curvature for plane curve

For plane curve in oriented plane OXY one can consider so called signed curvature (curvature with sign), i.e. curvature defined by the orientation of ordered pair $(\mathbf{v},\mathbf{a})$ which can be positive or negative. It is equal to the area of oriented parallelogram formed by velocity and acceleration vectors divided by the cube of the velocity:

$$k_{\text{sign}} = \frac{\det \begin{pmatrix} x_t & y_t \\ x_{tt} & y_{tt} \end{pmatrix}}{|\mathbf{v}(t)|^3} = \frac{x_t y_{tt} - x_{tt} y_t}{(x_t^2 + y_t^2)^{3/2}}, \quad k = |k_{\text{sign}}| \tag{48}$$

Signed curvature is defined by cross-product (area of oriented parallelogram). Usual curvature is defined by modulo of cross-product (area of non-oriented parallelogram).

Curvature coincides with signed curvature up to a sign.

Signed curvature is positive (coincides with curvature) if vectors $\mathbf{v},\mathbf{a}$ are oriented as ordered pair of basic vectors $\mathbf{e}_x,\mathbf{e}_y$:

$$k_{\text{sign}} = k, \text{ if rotation from } \mathbf{v} \text{ to } \mathbf{a} \text{ is counter clock wise.} \tag{49}$$

Signed curvature is negative (coincides with curvature up to a sign) if vectors $\mathbf{v},\mathbf{a}$ are oriented as ordered pair of basic vectors $\mathbf{e}_y,\mathbf{e}_x$:

$$k_{\text{sign}} = -k, \text{ if rotation from } \mathbf{v} \text{ to } \mathbf{a} \text{ is clock wise.} \tag{50}$$

We know that curvature does not depend on parameterisation. What about signed curvature? If curvature is equal to k then signed curvature can be equal to k or to $-k$. If we change parameterisation of curve $t \to -t$, i.e. roughly speaking move along the curve in opposite direction then signed curvature changes the sign (the vectors $(-\mathbf{v}, \mathbf{a})$ have orientation opposite to $(\mathbf{v}, \mathbf{a})$). Changing $t \to -t$ it means changing orientation of curve.

Signed curvature does not depend on parameterisation up to a sign. It changes a sign if we change orientation of the curve.

Before considering examples one additional remark:

Remark In the case if at the given point acceleration vector is orthogonal to the velocity vector then area of parallelogram is just equal to $|\mathbf{a}||\mathbf{v}|$ and curvature at the given point is equal to the length of acceleration divided by the square of velocity:

$$k\big|_{\mathbf{r}(t_0)} = \frac{|\mathbf{a}|}{|\mathbf{v}|^2} \qquad \text{in the case if } \mathbf{a} \text{ is orthogonal to } \mathbf{v} \text{ in the given point } \mathbf{r}(t_0) \tag{51}$$

Note that the condition that acceleration is orthogonal to velocity at the given point does not necessarily implies that it is orthogonal to the velocity at the all points in the vicinity of this point (like for natural parameter (see (28)). E.g. consider parabola in the parameterisation $x = t, y = t^2$. One can see that at the point $(0, 0)$ and only at this point acceleration is orthogonal to velocity.

Example 1 Consider ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{52}$$

$(a, b > 0)$ Consider the following parameterisation of this ellipse:

$$\mathbf{r} = \mathbf{r}(t)\colon \quad x = a\cos wt, y = b\sin wt, 0 \le t < \frac{2\pi}{w} \tag{53}$$

One can see that t is not natural parameter. So we cannot apply formula from definition. Make calculations according to Theorem:

$$\mathbf{v}(t) = \begin{pmatrix} -aw\sin wt, \\ wb\cos wt \end{pmatrix}, \quad \mathbf{a}(t) = \begin{pmatrix} -aw^2\cos wt, \\ -bw^2\sin wt \end{pmatrix} \tag{54}$$

The area of oriented parallelogram is equal to $S(\mathbf{v}, \mathbf{a}) = \det \begin{pmatrix} x_t, y_t \\ x_{tt}, y_{tt} \end{pmatrix} = abw^3$. It is positive. Signed curvature coincides with curvature. (This corresponds to the fact that the point moves counter clock-wise in (53), acceleration is directed in the interior of ellipse. The pair $(\mathbf{v}, \mathbf{a})$ has positive orientation. Hence curvature is equal to the signed curvature according to (49):

$$k = k_{\rm sign} = \frac{ab}{\sqrt{(a^2\sin^2 wt + b^2\cos^2 wt)^3}} \tag{55}$$

At the points $x = 0$ or $y = 0$ acceleration is orthogonal to velocity and curvature is equal to b/a^2 $(x = 0)$, a/b^2 $(y = 0)$.

If we change parameterisation then curvature remains the same..

Signed curvature $k_{\rm sign}$ remains the same under changing of parameterisation only up to the sign. Change orientation of the curve, i.e. consider parameterisation such that velocity vector moves clock wise, e.g. $t \to -t$ in (53). Then signed curvature $k_{\rm sign}$ will be negative. Point moves clock-wise, acceleration again is directed in the interior of ellipse. The pair $(\mathbf{v}, \mathbf{a})$ has negative orientation. Hence signed curvature becomes negative according to (50):

$$k_{\rm sign} = -k = -\frac{ab}{\sqrt{(a^2\sin^2 wt + b^2\cos^2 wt)^3}} \tag{56}$$

Example 2 Consider parabola $y = ax^2$. Assume that $a > 0$. Consider parameterisation $x = t, y = at^2$. t is not natural parameter (see footnote in the subsection "Natural parameterisation"). So we cannot apply formula from definition (32). Make calculations according to Theorem:

$$\mathbf{v}(t) = \begin{pmatrix} 1, \\ 2at \end{pmatrix}, \quad \mathbf{a}(t) = \begin{pmatrix} 0, \\ 2a \end{pmatrix} \tag{57}$$

The area of parallelogram is equal to $S(\mathbf{v}, \mathbf{a}) = \det \begin{pmatrix} x_t, y_t \\ x_{tt}, y_{tt} \end{pmatrix} = \det \begin{pmatrix} 1, 2at \\ 0, 2a \end{pmatrix} = 2a$. It is positive since $a > 0$. Hence curvature is equal to signed curvature and it is equal to

$$k = \frac{2a}{|\mathbf{v}|^3} = \frac{2a}{(1 + 4a^2t^2)^{\frac{3}{2}}}\,. \tag{58}$$

In particular at the point $(0, 0)$ where $t = 0$, $k = 2a$. Note that acceleration is orthogonal to velocity at the point (0.0) and only at this point!. Thus we can calculate curvature at the point (0.0) immediately by (51):

$$k = \frac{2a}{1} = 2a\,.$$

One can see that curvature coincides with signed curvature because the pair $(\mathbf{v}, \mathbf{a})$ is positive oriented:in standard parameterisation: $x = t, y = at^2$ velocity is directed to the right, acceleration- up.

If $a < 0$ then signed curvature is negative.

(Compare these calculations with calculation of curvature via touching circle in (37))

Example 3. Curve is a interval of line if and only if its curvature is equal to zero. Intuitively it is almost evident. Prove it. If curve is an interval of line then obviously one can consider parameterisation $x^i = a^i + b^i t$. Acceleration is equal to zero. Hence curvature is equal to zero. Prove inverse implication. If curvature is equal to zero then by (41) acceleration vector is parallel to velocity vector in arbitrary parameterisation. In particularly in natural parameterisation acceleration vector is equal to zero. Hence velocity vector is constant: $\frac{dx^i}{ds} = c^i$. Hence $x^i(s) = x_0^i + c^i s$. It is just equation of the line.

1.9 Integral of curvature along the plane curve.

Why so much attention to signed curvature for plane curve?

The signed curvature has the following beautiful property:

Let $\gamma\colon \mathbf{r} = \mathbf{r}(t)$ $a \leq t \leq b$ be plane curve on OXY. Let $k_{\text{sign}}(t)$ be a signed curvature for this curve.

Consider the following integral:

$$I_\gamma = \int_{t_1}^{t_2} k_{\text{sign}}(t) ds(t) dt = \int_{t_1}^{t_2} k_{\text{sign}}(t) |\mathbf{v}(t)| dt \tag{59}$$

One can easy to see that this integral is independent on reparameterisation up to the sign.

It turns out that this integral possesses the following property:

The value of this integral does not change if we deform the curve in way such that velocity vector in initial and final points remains unchanged.

Clarify our statement: Consider on the points of the curve $\mathbf{r}(t)$ the continuous function $\varphi(t)$ such that it is equal (up to $2\pi k$) to the angle between velocity vector $\mathbf{v}(t)$ and OX axis:

$$\varphi(t)\colon \tan\varphi(t) = \frac{y_t}{x_t}, t_1 \leq t \leq t_2 \tag{60}$$

E.g.for ellipse (53) $\varphi(t) = wt$. It changes from 0 till 2π. For parabola $x = t, y = at^2$, $\varphi(t) = \arctan 2at$. It changes from $-\frac{\pi}{2} = \varphi(-\infty)$ till $\frac{\pi}{2} = \varphi(\infty)$

Denote by $\varphi_1 = \varphi(t_1)$, $\varphi_2 = \varphi(t_2)$. Then the following remarkable identity holds:

$$I_\gamma = \int_{t_1}^{t_2} k_{\text{sign}}(t) ds(t) dt = \int_{t_1}^{t_2} \frac{d\varphi(t)}{dt} dt = \varphi_2 - \varphi_1 \tag{61}$$

Note that straightforward calculation often is difficult.

Proof When the result is formulated it is evident. Calculate the derivative of (60):

$$\frac{d\varphi}{dt} = \frac{d\tan\varphi}{dt}\frac{d\varphi}{d\tan\varphi} = \frac{d}{dt}\left(\frac{y_t}{x_t}\right)\frac{1}{1+\frac{y_t^2}{x_t^2}} = \frac{x_t y_{tt} - x_{tt} y_t}{x_t^2 + y_t^2}$$

We see that it is equal to integrand $k_{\text{sign}}(t)|\mathbf{v}(t)|$ in (61). ■

In the case if signed curvature $k_{\text{sign}}(t)$ does not change the sign (is positive or negative for all t) then this formula calculates integral of curvature up to the sign:

$$\int k(s) ds = |I_\gamma| \quad \text{if } k_{\text{sign}}(t) \text{ does not change the sign} \tag{62}$$

Consider two examples.

Example 1. Ellipse. Then use formulae of previous subsection (see (53)–(55)). Consider parameterisation $\varphi = wt$ Then

we come to the formula:

$$I_{\text{elipse}} = \int_0^{2\pi} \frac{ab}{a^2 \sin^2\varphi + b^2 \cos^2\varphi} d\varphi = 2\pi$$

For ellipse signed curvature has the same sign for all the points. Hence integral of curvature over ellipse is equal to $I_{\text{elipse}} = 2\pi$ (Try to calculate this integral in other way)

One can see that if γ is arbitrary closed convex curve, (i.e. its interior is convex domain) then integral of curvature along this curve will be equal to 2π.

Geometrical meaning: $k(s)\Delta s$ is equal to the rotation of the normal vector. We can interpret curvature as velocity of instantaneous rotation of normal vector (see in more detail the next subsection)

1.10 Frenèt frame for the curves in the plane

this subsection is compulsory only for MSC students

Let γ be plane curve– a curve in $\mathbf{E}^2$. Let $\mathbf{r}(s)$ be its natural parameterisation ($0 \leq s \leq s_0$). Consider velocity vector $\mathbf{v}(s)$. Acceleration vector $\mathbf{a}(s) = \frac{d\mathbf{v}}{ds}$ is orthogonal to velocity vector and its length is equal to the curvature $k(s)$.

Suppose that curvature is not equal to zero for every $s \in [0, s_0]$:

$$k(s) = \left|\mathbf{a}(s)\right| \neq 0 \tag{63}$$

Then acceleration vector $\mathbf{a}(s)$ defines unit vector $\mathbf{n}(s)$ which is orthogonal [4] to $\mathbf{v}$:

$$\frac{d\mathbf{v}(s)}{ds} = k(s)\mathbf{n}(s)\,. \tag{64}$$

Hence at every point of the curve where curvature $k(s) \neq 0$ we defined a basis (frame) $\{\mathbf{v}(s), \mathbf{n}(s)\}$ adjusted to the curve. This frame is orthonormal frame. It is called Frenèt frame.

This frame moves along the curve and rotates in the plane. One can show that

$$\frac{d\mathbf{n}(s)}{ds} = -k(s)\mathbf{v}(s). \tag{65}$$

This follows from (64). Indeed consider expansion of R.H.S. of the equation (65) with respect to the basis $\mathbf{v}(s), \mathbf{n}(s)$ in the plane:

$$\frac{d\mathbf{n}(s)}{ds} = \alpha(s)\mathbf{v}(s) + \beta(s)\mathbf{n}. \tag{66}$$

Multiplying both parts of this equation on $\mathbf{n}(s)$ we come to:

$$\mathbf{n} \cdot \frac{d\mathbf{n}}{ds} = \frac{1}{2}\frac{d}{ds}(\mathbf{n}\dot{\mathbf{n}}) = \frac{1}{2}\frac{d}{ds}(1) = 0 = \alpha(\mathbf{n} \cdot \mathbf{v}) + \beta(\mathbf{n} \cdot \mathbf{n}) = \beta$$

because $\mathbf{v}$ is orthogonal to $\mathbf{n}$: $(\mathbf{v} \cdot \mathbf{n}) = 0$. To prove that $\alpha(s) = -k(s)$ differentiate the identity $(\mathbf{v} \cdot \mathbf{n}) = 0$ by s:

$$0 = \frac{d}{ds}(\mathbf{v}\dot{\mathbf{n}}) = \frac{d\mathbf{v}}{ds}\mathbf{n} + \left(\mathbf{v} \cdot \frac{d\mathbf{n}}{ds}\right) = k(s)(\mathbf{n} \cdot \mathbf{n}) + \left(\mathbf{v} \cdot \frac{d\mathbf{n}}{ds}\right)$$

It follows from (66) that $\left(\mathbf{v} \cdot \frac{d\mathbf{n}}{ds}\right) = \alpha(s)$. Hence $\alpha(s) = \left(\mathbf{v} \cdot \frac{d\mathbf{n}}{ds}\right) = -k(s)$ We come to (65).

Equations (64), (65) are called Frenèt equations for Frenèt frame.

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[4] Note that there are two unit vectors which are orthogonal to $\mathbf{v}$. The direction of unit vector vector $\mathbf{n}$ in (64) is defined by the direction of acceleration $\mathbf{a}(s)$.

Their geometrical meaning is that

curvature $k(s)$ defines the speed of instantaneous rotation of Frenèt frame at the point $\mathbf{r}(s)$

To see it rewrite these equations in the following form:

$$\frac{d}{ds}\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\end{pmatrix} = \begin{pmatrix}k(s)\mathbf{n}(s)\\ -k(s)\mathbf{v}(s)\end{pmatrix} = \begin{pmatrix}0 & k(s)\\ -k(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\end{pmatrix} = k(s)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\end{pmatrix} \tag{67}$$

Consider Frenèt frame $(\mathbf{v}(s), \mathbf{n}(s))$ moving along the curve for $0 < s < a$.

Denote by $\varphi(s)$ be an angle of rotation: Then

$$\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\end{pmatrix} = \begin{pmatrix}\cos\varphi(s) & \sin\varphi(s)\\ -\sin\varphi(s) & \cos\varphi(s)\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \mathbf{n}\end{pmatrix}, \tag{68}$$

where we denote by $\begin{pmatrix}\mathbf{v}\\ \mathbf{n}\end{pmatrix}$ the Frenèt frame at the initial point $\mathbf{r}(0)$

Differentiate formula above along s:

$$\frac{d}{ds}\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\end{pmatrix} = \frac{d}{ds}\begin{pmatrix}\cos\varphi(s) & \sin\varphi(s)\\ -\sin\varphi(s) & \cos\varphi(s)\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \mathbf{n}\end{pmatrix} = \tag{69}$$

$$\begin{pmatrix}-\dot\varphi(s)\sin\varphi(s) & \dot\varphi(s)\cos\varphi(s)\\ -\dot\varphi(s)\cos\varphi(s) & -\dot\varphi(s)\sin\varphi(s)\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \mathbf{n}\end{pmatrix} = \tag{70}$$

$$\begin{pmatrix}0 & \dot\varphi(s)\\ -\dot\varphi(s) & 0\end{pmatrix}\cdot\begin{pmatrix}\cos\varphi(s) & \sin\varphi(s)\\ -\sin\varphi(s) & \cos\varphi(s)\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \mathbf{n}\end{pmatrix} = \dot\varphi(s)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\end{pmatrix} \tag{71}$$

Compare this for formula in (67). We see that:

$$\dot\varphi(s)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\end{pmatrix} = k(s)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\end{pmatrix} \tag{72}$$

i.e.

$$\dot\varphi(s) = k(s) \tag{73}$$

Curvature measures velocity of instantaneous rotation of the frame (Compare these considerations with considerations of previous subsection)

Comparing formulae above for Frenét frame rotation with formulae (59)–(61) we see that in particularly for for convex curve (closed curve such that its interior is convex domain) Frenèt frame makes rotation on the angle 2π.

It is easy to see that for an arbitrary closed oriented curve on the plane the rotation angle is equal to $2\pi n$, where n is equal to "winding number"

Remark In formulae above we assume that curvature is not equal zero at all s (see footnote to the formula (64)). If $k(s) = 0$ at some point $s = s_0$ then one needs *apriori* definition of direction of orthogonal vector. In the case if curve is in $\mathbf{E}^2$ then the direction of normal vector can be defined by orientation in $\mathbf{E}^2$ and orientation of curve: We choose $\mathbf{n}$ such that ordered pair $(\mathbf{v}, n)$ is positively oriented. It is easy to see that relative k_{sign} curvature considered in the subsections above can be defined as a proportionality coefficient between acceleration vector $\mathbf{a}(s)$ and normal unit vector $\mathbf{n}(s)$. If point moves along curve counter clockwise,

$$k_{\text{sign}}(s): \qquad \mathbf{a}(s) = k_{\text{sign}}(s)\mathbf{n}(s) \tag{74}$$

where $\mathbf{n}$ is chosen in the way that rotation from the vector $\mathbf{v}$ to vector $\mathbf{n}$ is counter clock wise. (Compare this definition with definition above.)

Remark It is easy to see that one can consider instead curvature (usual or signed) just a vector $\mathbf{A}(s)$ which is equal to acceleration vector in normal parameterisation. The vector $\mathbf{A}(s)$ is invariant (if we change $s \to a - s$ then $\mathbf{A}$ remains unchanged.) It is special case of second quadratic form.

1.11 Torsion

Considering higher derivatives of the curve one can consider Frenet frame for the curve in arbitrary n-dimensional Eucliden space $\mathbf{E}^n$.

Consider very briefly the case of $\mathbf{E}^3$.

Let $\mathbf{r}(s)$ be a curve in natural parameterisation in $\mathbf{E}^3$. Suppose that curvature is not equal to zero at all the points. In the same way as in(64) consider unit vector $\mathbf{n}(s)$ such that $\mathbf{a}(s) = k(s)\mathbf{n}(s)$, where $\mathbf{a}(s)$ is acceleration vector. Vectors $\mathbf{v}(s), \mathbf{a}(s)$ form orthonormal basis in osculating plane. Consider third unit vector $\mathbf{t}(s) = \mathbf{v}(s) \times \mathbf{a}(s)$. This vector is rothogonal to osculating plane. Three vectors $\{\mathbf{t}(s), \mathbf{v}(s), \mathbf{a}(s)\}$ form orthonormal basis in $\mathbf{E}^3$ adjusted to the curve. It is Frenet basis.

We have by definition of $\mathbf{n}(s)$

$$\frac{d}{ds}\mathbf{v}(s) = k(s)\mathbf{n}(s). \tag{75}$$

In the same way like in (65) one can deduce that

$$\frac{d}{ds}\mathbf{n}(s) = -k(s)\mathbf{v}(s) + \kappa(s)\mathbf{t}(s)\,. \tag{76}$$

Considering Frenet basis one can deduce the following analogue of equations (67):

$$\frac{d}{ds}\begin{pmatrix}\mathbf{v}(s)\\ \mathbf{n}(s)\\ \mathbf{t}(s)\end{pmatrix} = \begin{pmatrix} k(s)\mathbf{n}(s)\\ -k(s)\mathbf{v}(s) + \kappa(s)\mathbf{t}(s)\\ -\kappa(s)\mathbf{n}(s)\end{pmatrix} \tag{77}$$

Definition Proportionality coefficient $\kappa(s)$ in formulae (76), (77) is called a *torsion* of the curve.

In the same way as curve belongs to the line if and only if its curvature is equal to zero (see example in the subsection "Curvature"), one can see that torsion is equal to zero if and only if curve in the space belongs to plane.

2 Surfaces in E^3. First and Second quadratic forms. Gaussian and mean curvature.

2.1 Surfaces, tangent vectors

Let $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)$ be a parameterisation of surface in $\mathbf{E}^3$. (x, y, z are cartesian coordinates in $\mathbf{E}^3$) One can consider tangent vectors:

$$\frac{\partial \mathbf{r}(u,v)}{\partial u} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix}, \quad \frac{\partial \mathbf{r}(u,v)}{\partial v} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{pmatrix} \tag{78}$$

Later we often use shorter notations:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u,v)}{\partial u} = \begin{pmatrix} x_u(u,v) \\ y_u(u,v) \\ z_u(u,v) \end{pmatrix}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}(u,v)}{\partial v} = \begin{pmatrix} x_v(u,v) \\ x_v(u,v) \\ x_v(u,v) \end{pmatrix} \tag{79}$$

For example consider surface defined by the equation $z - F(x,y) = 0$. It can be parameterised:

$$\mathbf{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \begin{pmatrix} u \\ v \\ F(u,v) \end{pmatrix} \tag{80}$$

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u(u,v) \end{pmatrix}, \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v(u,v) \end{pmatrix} \tag{81}$$

Example Consider sphere of radius R:

$$\mathbf{r}(\theta,\varphi) = \begin{pmatrix} x(\theta,\varphi) \\ y(\theta,\varphi) \\ z(\theta,\varphi) \end{pmatrix} = \begin{pmatrix} R\sin\theta\cos\varphi \\ R\sin\theta\sin\varphi \\ R\cos\theta \end{pmatrix}. \tag{82}$$

$$\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix} \tag{83}$$

Tangent plane

Let p be a given point of the surface M. Consider the plane formed by the vectors which are adjusted to the point p and tangent to the surface M. We call this plane *plane tangent to M at the point p* and denote it by T_pM.

Let $\mathbf{r} = \mathbf{r}(t)$ be a curve belonging to the surface C, i.e. $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$. Let $p = \mathbf{r}(t_0)$ be any point on this curve. Then vector

$$\mathbf{r}_t = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}(u(t), v(t))}{dt} \tag{84}$$

belongs to the tangent plane T_pM.

Basis in tangent plane

Let $\mathbf{r} = \mathbf{r}(u, v)$ be a parameterisation of the surface M. Then for every point $p \in M$ one can consider a basis in the tangent plane T_pM adjusted to the parameters u, v. Every vector $\mathbf{X} \in T_pM$ can be expanded over this basis:

$$\mathbf{X} = X_u\mathbf{r}_u + X_v\mathbf{r}_v, \tag{85}$$

where X_u, X_v are coefficients, components of the vector $\mathbf{X}$.

The basis vector $\mathbf{r}_u \in T_pM$,is velocity vector for the curve $u = u_0 + t, v = v_0$, where (u_0, v_0) are coordinates of the point p. Respectively the basis vector $\mathbf{r}_v \in T_pM$,is velocity vector for the curve $u = u_0, v = v_0 + t$, where (u_0, v_0) are coordinates of the point p.

Note that for the vector (84) components X_u, X_v are equal to $X_u = u_t, X_v = v_t$ because

$$\mathbf{r}_t = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}(u(t), v(t))}{dt} = u_t\mathbf{r}_u + v_t\mathbf{r}_v \tag{86}$$

We begin to use condensed notations. In condensed notation instead denoting coordinates by (u, v) we often denote them by $u^\alpha = (u^1, u^2)$. Respectively we denote by

$$\mathbf{r}_\alpha = \frac{d\mathbf{r}}{du^\alpha}, \quad \mathbf{r}_u = \mathbf{r}_1, \mathbf{r}_v = \mathbf{r}_2$$

The formula (93) for tangent vector field will have the following appearance:

$$\mathbf{X} = X^\alpha\mathbf{r}_\alpha = X^1\mathbf{r}_1 + X^2\mathbf{r}_2, \quad (X^1 = X_u, X^2 = X_v) \tag{87}$$

The formula (84) will have the appearance:

$$\frac{d\mathbf{r}}{dt} = u_t^\alpha r_\alpha = \frac{du^\alpha}{dt}\mathbf{r}_\alpha$$

When using condensed notations we usually omit explicit summation symbols. E.g. we write $u^\alpha \mathbf{r}_\alpha$ instead $\sum_{i=1}^2 u^\alpha \mathbf{r}_\alpha$ or $u^1\mathbf{r}_1 + u^2\mathbf{r}_2$

One can consider also differentials $du^\alpha = (du^1, du^2)$:

$$du^\alpha(\mathbf{r}_\beta) = \delta_\beta^\alpha: \quad du^1(\mathbf{r}_1) = du^2(\mathbf{r}_2) = 1,\ du^1(\mathbf{r}_2) = du^2(\mathbf{r}_1) = 0 \tag{88}$$

2.2 Reparameterisation in condensed notations

Study how formulae above change if we change parameterisation:

Let $u^\alpha = u^\alpha(\xi^p)$ $(\alpha = 1, 2, , p = 1, 2)$. It is condensed notation for changing of parameters u^1, u^2 on new parameters ξ^1, ξ^2: $u^1 = u^1(\xi^1, \xi^2), u^2 = u^2(\xi^1.\xi^2)$.

Then for (88)

$$du^a = \frac{\partial u^\alpha}{\partial \xi^p} d\xi^p\,, \left(du^1 = \frac{\partial u^1}{\partial \xi^1} d\xi^1 + \frac{\partial u^1}{\partial \xi^2} d\xi^2\right), \left(du^2 = \frac{\partial u^2}{\partial \xi^1} d\xi^1 + \frac{\partial u^2}{\partial \xi^2} d\xi^2\right) \tag{89}$$

For basis vectors:

$$\mathbf{r}_a = \xi_p^a \mathbf{r}_p = \frac{\partial \xi^p(u)}{\partial u^\alpha}\frac{\partial \mathbf{r}}{\partial \xi^p},\ \text{i.e.} \tag{90}$$

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}_1}{\partial u^1} = \frac{\partial \xi^1(u)}{\partial u^1}\frac{\partial \mathbf{r}}{\partial \xi^1} + \frac{\partial \xi^2(u)}{\partial u^1}\frac{\partial \mathbf{r}}{\partial \xi^2},\ \mathbf{r}_2 = \frac{\partial \mathbf{r}_1}{\partial u^2} = \frac{\partial \xi^1(u)}{\partial u^2}\frac{\partial \mathbf{r}}{\partial \xi^1} + \frac{\partial \xi^2(u)}{\partial u^2}\frac{\partial \mathbf{r}}{\partial \xi^2}$$

and for tangent vectors:

$$\mathbf{X} = X^\alpha \mathbf{r}_\alpha = X^\alpha \xi_\alpha^p \mathbf{r}_p = (X^\alpha \xi_\alpha^p)\,\mathbf{r}_\alpha \Rightarrow X^p = \xi_\alpha^p X^\alpha, \text{e.g.}\, X^{1'} = \xi_1^{1'} X^1 + \xi_2^{1'} X^2 \tag{91}$$

For curve:

$$\frac{d\mathbf{r}}{dt} = u_t^\alpha r_\alpha = u_t^\alpha \xi_\alpha^p \mathbf{r}_p \tag{92}$$

2.3 Internal and external coordinates of tangent vector

Look again on the formulae (93), (87). Denote $X_u = a$, $X_v = b$

$$\mathbf{X} = X_u\mathbf{r}_u + X_v\mathbf{r}_v = a\mathbf{r}_u + b\mathbf{r}_v = a\begin{pmatrix} x_u(u,v) \\ y_u(u,v) \\ z_u(u,v)\end{pmatrix} + b\begin{pmatrix} x_v(u,v) \\ x_v(u,v) \\ x_v(u,v)\end{pmatrix} = \begin{pmatrix} ax_u(u,v) + bx_v(u,v) \\ ay_u(u,v) + by_v(u,v) \\ az_u(u,v) + bz_v(u,v)\end{pmatrix} \tag{93}$$

(a, b) can be considered as *internal coordinates of the tangent vector* $\mathbf{X}$. Coordinates of the vector $\mathbf{X}$ in the ambient space

$$(ax_u(u,v) + bx_v(u,v), ay_u(u,v) + by_v(u,v), az_u(u,v) + bz_v(u,v))$$

can be considered as external coordinates of the tangent vector $\mathbf{X}$. Ant living on the surface deals with the vector $\mathbf{X}$ in terms of coordinates (a, b). External observer which contemplates the surface embedded in three-dimensional space deals with vector $\mathbf{X}$ as with vector with external coordinates $(ax_u(u,v) + bx_v(u,v), ay_u(u,v) + by_v(u,v), az_u(u,v) + bz_v(u,v))$.

2.4 First Quadratic Form

Definition

Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface embedded in $\mathbf{E}^3$.

First quadratic form defines length of the tangent vector to the surface in internal coordinates and distance between points of the surface.

The first quadratic form at the point $\mathbf{r} = \mathbf{r}(u, v)$ is defined by symmetric matrix:

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix}, \quad G_{\alpha\beta} = (\mathbf{r}_\alpha, \mathbf{r}_\beta) \tag{94}$$

where $(\, , \,)$ is a scalar product:

$$G = G_{\alpha\beta} du^\alpha du^\beta = G_{11} du^2 + 2G_{12} du dv + G_{22} dv^2 \tag{95}$$

Consider a vector $\mathbf{X} = X^\alpha \mathbf{r}_\alpha = a\mathbf{r}_u + b\mathbf{r}_v$ tangent to the surface M.

The square of the length $|\mathbf{X}|$ of this vector

$$|\mathbf{X}|^2 = (\mathbf{X}, \mathbf{X}) = < a\mathbf{r}_u + b\mathbf{r}_v, a\mathbf{r}_u + b\mathbf{r}_v > = a^2(\mathbf{r}_u, \mathbf{r}_u) + 2ab(\mathbf{r}_u, \mathbf{r}_v) + b^2(\mathbf{r}_v, \mathbf{r}_v) \tag{96}$$

It is just equal to the value of the first quadratic form on this tangent vector:

$$G(\mathbf{X}, \mathbf{X}) = \begin{pmatrix} a, & b \end{pmatrix} \cdot \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = G_{11} a^2 + 2G_{12} ab + G_{22} b^2 \tag{97}$$

External observer (person living in ambient space $\mathbf{E}^3$) calculate the length of the tangent vector using formula (96). An ant living on the surface calculate length of this vector in internal coordinates using formula (97). External observer deals with external coordinates of the vector, ant on the surface with internal coordinates.

If $\mathbf{X}, \mathbf{Y}$ are two tangent vectors in the tangent plane T_pC then $G(\mathbf{X}, \mathbf{Y})$ at the point p is equal to scalar product of vectors $\mathbf{X}, \mathbf{Y}$:

$$(\mathbf{X}, \mathbf{Y}) = (X^1\mathbf{r}_1 + X^2\mathbf{r}_2, Y^1\mathbf{r}_1 + Y^2\mathbf{r}_2) = \tag{98}$$
$$X^1(\mathbf{r}_1, \mathbf{r}_1)Y^1 + X^1(\mathbf{r}_1, \mathbf{r}_2)Y^2 + X^2(\mathbf{r}_2, \mathbf{r}_1)Y^1 + X^2(\mathbf{r}_2, \mathbf{r}_2)Y^2 =$$
$$X^\alpha(\mathbf{r}_\alpha, \mathbf{r}_\beta)Y^\beta = X^\alpha G_{\alpha\beta}Y^\beta = G(\mathbf{X}, \mathbf{Y})$$

Remark We identify quadratic forms and corresponding symmetric bilinear forms [5]

First quadratic form and length of the curve

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \le t \le b$ be a curve on the surface.

The first quadratic form measures the length of velocity vector at every point of this curve. Thus we come to the formula for length of the curve.

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to

$$\mathbf{v} = \mathbf{X} = \xi\mathbf{r}_u + \eta\mathbf{r}_v \text{where } \xi = u_t, \eta = v_t: \quad \mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = u_t\mathbf{r}_u + v_t\mathbf{r}_v\,.$$

The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)|dt = \int_a^b \sqrt{(\mathbf{v}(t), \mathbf{v}(t))}dt = \int_a^b \sqrt{(u_t\mathbf{r}_u + v_t\mathbf{r}_v, u_t\mathbf{r}_u + v_t\mathbf{r}_v)}dt = \tag{100}$$
$$\int_a^b \sqrt{(\mathbf{r}_u, \mathbf{r}_u)u_t^2 + 2(\mathbf{r}_u, \mathbf{r}_v)u_tv_t + (\mathbf{r}_v, \mathbf{r}_v)v_t^2}d\tau =$$
$$\int_a^b \sqrt{G_{11}u_t^2 + 2G_{12}u_tv_t + G_{22}v_t^2}dt \tag{101}$$

An external observer will calculate the length of the curve using (22). An ant living on the surface calculate length of the curve via first quadratic form using (101): first quadratic form defines Riemannian metric on the surface:

$$ds^2 = G_{ik}du^idu^k = G_{11}du^2 + 2G_{12}dudv + G_{22}dv^2 \tag{102}$$

[5] Bilinear symmetric form $B(\mathbf{X}, \mathbf{Y}) = B(\mathbf{Y}, \mathbf{X})$ defines quadratic form $Q(\mathbf{X}) = B(\mathbf{X}, \mathbf{X})$. Quadratic form satisfies the condition $Q(\lambda\mathbf{X}) = \lambda^2 Q(\mathbf{X})$ and so called parallelogram condition

$$Q(\mathbf{X} + \mathbf{Y}) + Q(\mathbf{X} - \mathbf{Y}) = 2Q(\mathbf{X}) + 2Q(\mathbf{Y}) \tag{99}$$

Invariance of first quadratic form

We give above an invariant definition of the first quadratic form. Double check that it is reparameterisation invariant: Let ξ^1, ξ^2 be new parameters: $u^\alpha = u^\alpha(\xi^p)$.

Let first quadratic form G is equal to $du^\alpha G_{\alpha\beta} du^\beta$ in parameters (u^1, u^2) and it is equal to $d\xi^p G'_{pq} d\xi^q$ in new parameters (ξ^1, ξ^2), where $G_{\alpha\beta} = (\mathbf{r}_\alpha, \mathbf{r}_\beta) = (\frac{\partial \mathbf{r}}{\partial u^\alpha}, \frac{\partial \mathbf{r}}{\partial u^\beta})$ and respectively $G'_{pq} = (\mathbf{r}_p, \mathbf{r}_q) = (\frac{\partial \mathbf{r}}{\partial \xi^p}, \frac{\partial \mathbf{r}}{\partial \xi^q})$. We have to check that $du^\alpha G_{\alpha\beta} du^\beta = d\xi^p G'_{pq} d\xi^q$. Using (89) and (90) we see that

$$du^\alpha G_{\alpha\beta} du^\beta = du^\alpha (\mathbf{r}_\alpha, \mathbf{r}_\beta) du^\beta = d\xi^p (\mathbf{r}_p, \mathbf{r}_q) d\xi^q = d\xi^p G'_{pq} d\xi^q \quad \blacksquare \tag{103}$$

2.5 Second Quadratic Form

First quadratic form and corresponding symmetric bilinear form measure length of tangent vector and scalar product of tangent vectors and length of the curve.

Now we define the second quadratic form which measures curvature. For curves in $\mathbf{E}^n$ we define curvature via acceleration and velocity vectors. For different curves beginning at the giving point curvature is different. On the other hand it has to depend on second derivatives.

We give a formal definition for second quadratic form and show that it is reparameterisation invariant. Then we will reveal its geometrical meaning.

Definition–Proposition

Let M be a surface given by parameterisation $\mathbf{r} = \mathbf{r}(u, v)$. Consider at every point of surface the following form:

$$\mathbf{A} = (\mathbf{n}, \mathbf{r}_{uu}) du^2 + 2(\mathbf{n}, \mathbf{r}_{uv}) du dv + (\mathbf{n}, \mathbf{r}_{vv}) dv^2 \tag{104}$$

where $\mathbf{n}$ is a unit normal vector and

$$\mathbf{r}_{uu} = \frac{\partial^2 \mathbf{r}}{\partial^2 u}, \ \mathbf{r}_{uv} = \frac{\partial^2 \mathbf{r}}{\partial u \partial v}, \ \mathbf{r}_{vv} = \frac{\partial^2 \mathbf{r}}{\partial^2 v}$$

This expression defines the *second quadratic form* and corresponding bilinear form on tangent vectors:

$$\mathbf{A} = A_{uu} du^2 + 2A_{uv} du dv + A_{vv} dv^2 \tag{105}$$

$$\mathbf{A}(\mathbf{X}, \mathbf{X}) = A_{uu} a^2 + 2A_{uv} ab + A_{vv} b^2 \text{ if } \mathbf{X} = a\mathbf{r}_u + b\mathbf{r}_v \tag{106}$$

In condensed notations:

$$\mathbf{A} = du^\alpha A_{\alpha\beta} du^\beta = du^\alpha (\mathbf{n}, \mathbf{r}_{\alpha\beta}) du^\alpha \tag{107}$$

$$\mathbf{A}(\mathbf{X}, \mathbf{Y}) = X^\alpha A_{\alpha\beta} Y^\beta, \text{ where } \mathbf{X} = X^\alpha \mathbf{r}_\alpha, \mathbf{Y} = Y^\alpha \mathbf{r}_\alpha \tag{108}$$

$$A_{\alpha\beta} = (\mathbf{n}, \mathbf{r}_{\alpha\beta}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{n}, \mathbf{r}_{uu}) & (\mathbf{n}, \mathbf{r}_{uv}) \\ (\mathbf{n}, \mathbf{r}_{vu}) & (\mathbf{n}, \mathbf{r}_{vv}) \end{pmatrix}$$

We have to prove that expression (104) indeed defines quadratic and corresponding bilinear form (108), i.e. (104) is invariant under changing of parameterisation.

Remark The proof of reparameterisation invariance for first quadratic form was double checking. First quadratic form was invariant *apriori*. In the case of second quadratic form the invariance does not follow *apriori* from the definition (104)

Let $\mathbf{A} = du^\alpha A_{\alpha\beta} du^\beta$ be an appearance of second quadratic form in coordinates (u^1, u^2). Let $d\xi^p A'_{pq} d\xi^q$ be an appearance of second quadratic form in new coordinates (ξ^1, ξ^2). Here $A_{\alpha\beta} = (\mathbf{n}, \mathbf{r}_{\alpha\beta}) = (\mathbf{n}, \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta})$ and respectively $A'_{pq} = (\mathbf{r}_p, \mathbf{r}_q) = (\frac{\partial^2 \mathbf{r}}{\partial \xi^p \partial \xi^q}, \frac{\partial \mathbf{r}}{\partial \xi^q})$. We have to check that $\mathbf{A} = du^\alpha A_{\alpha\beta} du^\beta = d\xi^p A'_{pq} d\xi^q$.

Let ξ^1, ξ^2 be new parameters: $u^\alpha = u^\alpha(\xi^p)$.

Using chain rule calculate $A_{pq} = (\mathbf{n}, \frac{\partial^2 \mathbf{r}}{\partial \xi^p \xi^q})$:

$$\frac{\partial^2 \mathbf{r}}{\partial \xi^p \partial \xi^q} = \frac{\partial}{\partial \xi^p} \left(\frac{\partial u^\beta}{\partial \xi^q} \frac{\partial \mathbf{r}}{\partial u^\beta} \right) = \frac{\partial^2 u^\beta}{\partial \xi^p \partial \xi^q} \frac{\partial \mathbf{r}}{\partial u^\beta} + \frac{\partial u^\alpha}{\partial \xi^p} \frac{\partial u^\beta}{\partial \xi^q} \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta}$$

Now: simple but important observation: in the last formula the first term in the RHS which possesses second derivatives of reparameterisation, $\frac{\partial^2 u^a}{\partial \xi^p \partial \xi^q}$ is proportional to tangent vector $\frac{\partial \mathbf{r}}{\partial u^\alpha}$. Hence its scalar product with normal vector $\mathbf{n}$ is equal to zero:

$$\left(\mathbf{n}, \frac{\partial^2 \mathbf{r}}{\partial \xi^p \partial \xi^q} \right) = \frac{\partial^2 u^\beta}{\partial \xi^p \partial \xi^q} \underbrace{\left(\mathbf{n}, \frac{\partial \mathbf{r}}{\partial u^\beta} \right)}_{\text{vanishes}} + \frac{\partial u^\alpha}{\partial \xi^p} \left(\mathbf{n}, \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} \right) \frac{\partial u^\beta}{\partial \xi^q}$$

$$A_{pq} = (\mathbf{n}, \mathbf{r}_{pq}) = \left(\mathbf{n}, u^\alpha_{pq} \mathbf{r}_a + u^\alpha_p u^\beta_q \mathbf{r}_{\alpha\beta} \right) = \left(\mathbf{n}, u^\alpha_p u^\beta_q \mathbf{r}_{\alpha\beta} \right) = u^\alpha_p A_{\alpha\beta} u^\beta_q \tag{109}$$

(We use notations: $u^a_p = \frac{\partial u^\alpha}{\partial \xi^p}$, $u^\alpha_{pq} = \frac{\partial^2 u^\alpha}{\partial \xi^p \partial \xi^q}$)

The formula above establishes the transformation of components of second quadratic form under changing of parameters.

Using (89) we see that

$$du^\alpha A_{\alpha\beta} du^\beta = d\xi^p u_p^\alpha A_{\alpha\beta} u_q^\beta d\xi^q = d\xi^p A'_{pq} u_q^\beta d\xi^q \blacksquare \qquad (110)$$

We came above to the notion of the first quadratic form calculating length of vectors tangent to the surface and length of the curve on the surface.

What about to calculate acceleration and curvature. For curves acceleration defines curvature, at least in normal parameterisation. In arbitrary parameterisation curvature is defined by velocity and acceleration vectors (see (41))).

Our task is to define a curvature on the points of surface.

Note that different curves starting at this point have different curvatures. Sure curvature depends on direction of the curve: (consider e.g. cylinder.)

But it is not the end of the story. Consider curves starting at the given point which are tangent to the same vector $\mathbf{X}$:

$$\mathbf{r}(t) = \mathbf{r}(u(t), v(t)) = r(u_0, v_0) + t\mathbf{X} + \dots \qquad \mathbf{v}(u_0, v_0) = \frac{d\mathbf{r}(t)}{dt}\Big|_{t=0} \qquad (111)$$

Curvature depends on second derivatives. Even fixing tangent vector we do not fix curvature of the curve.

2.6 Second quadratic form and curvature of normal sections

Let p be an arbitrary point of the surface M. Let $\mathbf{n}$ be normal vector to the surface M at the point p, i.e. $\mathbf{n}$ is orthogonal to the surface at the point p and the length of $\mathbf{n}$ is equal to 1:

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \qquad (112)$$

Definition

A plane which passes through the point p and possesses the vector $\mathbf{n}$ is called normal plane at the point p.

An intersection of normal plane with surface gives a curve. This curve is called *normal section at the point p*

Let $\mathbf{n}$ be a normal vector at the point p and $\mathbf{X}$ be a tangent vector at this point. Consider plane spanned by vectors $\mathbf{n}, \mathbf{X}$. This will be normal plane which possesses vector $\mathbf{X}$. Intersection of the normal plane with surface will be normal section. This normal section we denote by $l_{\mathbf{X}}(p)$. $\mathbf{X}$ is tangent vector of normal section $l_{\mathbf{X}}(p)$ at the point p.

We will calculate now the curvature of normal section via second and first quadratic forms.

Proposition *Second quadratic form measures curvature of normal sections. The curvature $k(l_{\mathbf{X}})$ of normal section $l_{\mathbf{X}}(p)$ at the point p is given up to the sign by the following formula:*

$$k(l_{\mathbf{X}}) = \frac{\mathbf{A}(\mathbf{X},\mathbf{X})}{G(\mathbf{X},\mathbf{X})}, \tag{113}$$

where $\mathbf{A}$ is the the second quadratic form at the point p and G is the first quadratic form at the point p.

In particularly if one chooses parameterisation such that $|\mathbf{X}| = \mathbf{1}$ then

$$k(l_{\mathbf{X}}) = \mathbf{A}(\mathbf{X},\mathbf{X}), \tag{114}$$

Remark Here we again can consider signed curvature as in subsection 1.7. As it was mentioned before unit vector $\mathbf{n}$ is decided up to a sign. Hence second quadratic form $\mathbf{A}(\mathbf{X},\mathbf{X})$ is defined up to the sign too.

Fixing the direction of $\mathbf{n}$ (one can do it using orientation or in other way) fixes the sign of curvature for a normal section.

Proof.

Before going to calculations note that right hand side of the formula in proposition does not depend on the length of the vector $\mathbf{X}$, i.e. it is invariant with respect to different parameterisation of normal section: we change $\mathbf{X} \to a\mathbf{X}$ then numerator and denominator are multiplied on a^2.

Let $\mathbf{r} = \mathbf{r}(u(t), v(t)$ be parametrisation of normal section $l_{\mathbf{X}}$ at the point p.

Let $\mathbf{v}$ be velocity vector and $\mathbf{a}$ be acceleration vector of the normal section $l_{\mathbf{X}}$ at the point initial point p. Velocity vector is proportional to tangent vector $\mathbf{X}$. If $\mathbf{r} = r(u(t), v(t))$ is parametric equation of normal curve then $\mathbf{v} = \mathbf{X} = \mathbf{r}_u u_t + \mathbf{r}_v v_t$ at the initial point p. Without loss of generality suppose that $\mathbf{v} = \mathbf{X}$. (If $\mathbf{v} \neq X$, we change parameter $t \to at$, then $\mathbf{v} \to a\mathbf{v}$)

Calculate acceleration at the point p. Normal section belongs to the normal plane formed by normal vectors $\mathbf{n}$ and velocity vector $\mathbf{v} = \mathbf{X}$. Hence acceleration vector is linear combination of these vectors:

$$\mathbf{a} = \mathbf{a}_{\text{perpendic}} + \mathbf{a}_{\text{parallel}} = L\mathbf{n} + b\mathbf{X} \tag{115}$$

Calculate curvature at the point p. Curvature k is equal to the modulus of the vector $\mathbf{v}\times\mathbf{a}$ divided by the cube of the modulus of the velocity vector $k=\frac{|\mathbf{v}\times\mathbf{a}|}{|v|^3}$ (see (45)). On the other hand according the formula above

$$\mathbf{v}\times\mathbf{a}=\mathbf{v}\times L\mathbf{n}+b\mathbf{X}=L\mathbf{X}\times\mathbf{n},\ \text{and}\ |\mathbf{v}\times\mathbf{a}|=L|\mathbf{X}|\,.$$

Hence

$$k=\frac{|\mathbf{v}\times\mathbf{a}|}{|v|^3}=\frac{|\mathbf{a}_{\text{perpendic}}|}{|\mathbf{X}|^3}=\frac{L}{|X|^2}$$

It remains to calculate coefficient L in the expansion (115) of acceleration vector. Show that it is equal to the value of second quadratic form on the velocity vector $\mathbf{X}$. Take a scalar product of $\mathbf{a}$ on the unit vector $\mathbf{n}$: $(\mathbf{X},\mathbf{n})=0$, hence $(\mathbf{a},\mathbf{n})=(L\mathbf{n}+b\mathbf{X},\mathbf{n})=L$. We come to

$$L=(\mathbf{a},\mathbf{n})=\left(\frac{d^2\mathbf{r}}{dt^2},\mathbf{n}\right)=\left(\frac{d}{dt}\left(\mathbf{r}_u u_t+\mathbf{r}_v v_t\right),\mathbf{n}\right)=$$

$$\left(\underbrace{(\mathbf{r}_u u_{tt}+\mathbf{r}_v v_{tt})}_{\text{vector tangent to the surface}}+\left(\mathbf{r}_{uu}(u_t)^2+2\mathbf{r}_{uv}u_t v_t+\mathbf{r}_{vv}(v_t)^2\right),\mathbf{n}\right)=$$

$$\left(\left(\mathbf{r}_{uu}(u_t)^2+2\mathbf{r}_{uv}u_t v_t+\mathbf{r}_{vv}(v_t)^2\right),\mathbf{n}\right)=\left(\mathbf{r}_{uu},\mathbf{n}\right)u_t^2+2\left(\mathbf{r}_{uv},\mathbf{n}\right)u_t v_t+\left(\mathbf{r}_{vv},\mathbf{n}\right)v_t^2=$$

$$A_{uu}u_t^2+A_{uv}u_t v_t+A_{vv}v_t^2=\mathbf{A}(\mathbf{X},\mathbf{X})\,.$$

Hence we see that curvature of normal section is equal to

$$k=\frac{L}{|\mathbf{X}|^2}=\frac{\mathbf{A}(\mathbf{X},\mathbf{X})}{G(\mathbf{X},\mathbf{X})}$$

because $|\mathbf{X}|^2=G(\mathbf{X},\mathbf{X})$, This is just (113).

2.7 Shape operator and Gaussian and Mean Curvatures

Consider first and second quadratic forms $G(\mathbf{X},\mathbf{X})$, $A(\mathbf{X},\mathbf{X})$ for arbitrary point $\mathbf{r}(u,v)$ of the surface and arbitrary tangent vector $\mathbf{X}$. We proved (see Proposition (113), (114)) that

$$\frac{A(\mathbf{X},\mathbf{X})}{G(\mathbf{X},\mathbf{X})}=k(l_{\mathbf{X}})(\text{curvature of the normal curve } l_{\mathbf{X}} \text{ at the initial point})$$

It depends on the direction of vector $\mathbf{X}$ and does not depend on its value. Considering a circle $|\mathbf{X}| = 1$ we see that in a general case there are two directions such that in one direction curvature $k_{\max}$ is *maximal* and in the other direction curvature $k_{\min}$is *minimal*:

$$k_{\min} \leq k(l_{\mathbf{X}}) \leq k_{\max} \tag{116}$$

Definition Product of maximal and minimal curvatures is called *Gaussian curvature.* Their sum is called *Mean curvature*:

$$K = k_{\max} k_{\min} = \frac{1}{r_1} \cdot \frac{1}{r_2}, \qquad H = k_{\max} + k_{\min} = \frac{1}{r_1} + \frac{1}{r_2} \tag{117}$$

Note that here we consider signed curvature (see remark after the formula (114). If we change the direction of normal vector $\mathbf{n}$ then all curvatures $k_{\min}, k_{l_{\mathbf{X}}}, k_{\max}$ change the sign K Gaussian curvature $k_{\min} k_{\max}$ remains unchanged. Mean curvature change the sign.

Consider *shape operator* $S(\mathbf{X})$ defined by the following way:

$$S(\mathbf{X}): \quad A(\mathbf{X}, \mathbf{X}) = G(\mathbf{X}, S(\mathbf{X})) \tag{118}$$

For corresponding bilinear forms

$$S(\mathbf{X}): A(\mathbf{Y}, \mathbf{X}) = G(\mathbf{Y}, S(\mathbf{X})), \text{for an arbitrary tangent vector } \mathbf{Y} \tag{119}$$

$$S = G^{-1} \cdot A$$

Theorem *Eigenvectors of shape operator define directions in which curvature is maximal and minimal. Eigenvalues of shape operator are maximal and minimal curvatures.*: If vector $\mathbf{X}_1$ defines the direction in which curvature of normal section is maximal: $k(l_{\mathbf{X}_1}) = k_{\max}$ and vector $\mathbf{X}_2$ defines the direction in which curvature of normal section is minimal: $k(l_{\mathbf{X}_2}) = k_{\min}$then

$$k_{l_{\mathbf{X}_1}} = k_{\max}, \quad k_{l_{\mathbf{X}_2}} = k_{\min}, \quad S\mathbf{X}_1 = k_{\max}\mathbf{X}_1, \ S\mathbf{X}_2 = k_{\max}\mathbf{X}_2 \tag{120}$$

Vectors $\mathbf{X}_1$, $\mathbf{X}_2$ are orthogonal (if $k_1 \neq k_2$)

Gaussian curvature is a product of maximal and minimal curvatures. It is equal to the determinant of Shape operator, i.e.to the ratio of the determinants of second and first quadratic forms:

$$K = k_{\max} k_{\min} = \det S = \det(G^{-1} \cdot A) = \frac{\det A}{\det G} \tag{121}$$

Mean curvature is a sum of maximal and minimal curvatures. It is equal to the Trace of the Shape operator:

$$K = k_{\max} + k_{\min} = \mathrm{Tr}S = \mathrm{Tr}(G^{-1} \cdot A) \tag{122}$$

Give a short proof of this Theorem.

Let $\mathbf{X}_1 = \begin{pmatrix} a_{max} \\ b_{max} \end{pmatrix}$ be a vector such that curvature of normal section $l_{\mathbf{X}_1}$ is maximal. Respectively let $\mathbf{X}_2 == \begin{pmatrix} a_{min} \\ b_{min} \end{pmatrix}$ be a vector such that curvature of normal section $l_{\mathbf{X}_2}$ is minimum. We consider components of the vector in the basis $\mathbf{r}_u, \mathbf{r}_v$ (e.g. $\mathbf{X}_{max} = a_{max}\mathbf{r}_u + b_{max}\mathbf{r}_v$)

Consider matrices $\mathbf{A} = \begin{pmatrix} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{pmatrix}$ and $G = \begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix}$ of second and first quadratic forms.

Then

$$\mathbf{A}\mathbf{X}_1 = k_{max}G\mathbf{X}_1, \text{ i.e. } \begin{pmatrix} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{pmatrix}\begin{pmatrix} a_{max} \\ b_{max} \end{pmatrix} = k_{max}\begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix}\begin{pmatrix} a_{max} \\ b_{max} \end{pmatrix} \tag{123}$$

and

$$\mathbf{A}\mathbf{X}_2 = k_{min}G\mathbf{X}_1, \text{ i.e. } \begin{pmatrix} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{pmatrix}\begin{pmatrix} a_{min} \\ b_{min} \end{pmatrix} = k_{min}\begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix}\begin{pmatrix} a_{min} \\ b_{min} \end{pmatrix} \tag{124}$$

Indeed k_{max} (k_{min}) is maximum (minimum) value of the function

$$A(a,b) = \mathbf{A}(\mathbf{X},\mathbf{X}) = A_{uu}a^2 + 2A_{uv}ab + A_{vv}b^2$$

subject to the condition that

$$G(a,b) = G_{uu}a^2 + 2G_{uv}ab + G_{vv}b^2 \equiv 1$$

Standard Lagrange multipliers consderation gives: $\frac{\partial A}{\partial a} = \lambda\frac{\partial G}{\partial a}$, $\frac{\partial A}{\partial b} = \lambda\frac{\partial G}{\partial b}$, i.e.

$$A_{uu}a + A_{uv}b = \lambda\left(G_{uu}a + G_{uv}b\right),\ A_{uv}a + A_{vv}b = \lambda\left(G_{uv}a + G_{vv}b\right)$$

It is just formulae (123), (124) above.

It follows from these relations that that $\mathbf{X}_1, \mathbf{X}_2$ are eigenvectors of shape operator $G^{-1}\mathbf{A}$:

$$\begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix}^{-1} \cdot \begin{pmatrix} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{pmatrix}\begin{pmatrix} a_{max} \\ b_{max} \end{pmatrix} = k_{max}\begin{pmatrix} a_{max} \\ b_{max} \end{pmatrix}$$

and respectively

$$\begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix}^{-1} \cdot \begin{pmatrix} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{pmatrix} \begin{pmatrix} a_{min} \\ b_{min} \end{pmatrix} = k_{min} \begin{pmatrix} a_{min} \\ b_{min} \end{pmatrix}$$

Orthogonality of eigenvectors $\mathbf{X}_1, \mathbf{X}_2$ (in the case if $k_{max} \neq k_{min}$) follows immediately from the relation:

$$k_{max}(\mathbf{X}_1, \mathbf{X}_2) = G(S\mathbf{X}_1, \mathbf{X}_2) = \mathbf{A}(\mathbf{X}_1, \mathbf{X}_2) = \mathbf{A}(\mathbf{X}_2, \mathbf{X}_1) = G(S\mathbf{X}_2, \mathbf{X}_1) = k_{min}(\mathbf{X}_2, \mathbf{X}_1)$$

(It is general property of eigenvectors of symmetric operators.)

Note It is instructive to consider the function $\det(1 + zS)$, where z is formal parameter. This function is quadratic polynomial in z and coefficients are just mean and Gaussian curvature:

$$\det(1 + zS) = 1 + Hz + Kz^2 \tag{125}$$

If M is a compact surface in $\mathbf{E}^3$ then one can consider remarkable polynomial:

$$P_M(z) = \int_M \det(1 + zS)\sqrt{g}d^2x \tag{126}$$

Remarkable formulae are related with this object (see Appendix "Tubes")

2.8 Calculations of Gaussian curvature and Mean curvatures

First of all calculate curvatures for the general case when a surface C is defined by the equation $z - F(x, y) = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(u, v): \quad \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \tag{127}$$

1. **Calculation of first quadratic form**

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} \tag{128}$$

,

$$(\mathbf{r}_u, \mathbf{r}_u) = 1 + F_u^2, \quad (\mathbf{r}_u, \mathbf{r}_v) = F_u F_v, \quad (\mathbf{r}_v, \mathbf{r}_v) = 1 + F_v^2$$

and first quadratic form (95) is equal to

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} \tag{129}$$

$$G = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix}, \quad ds^2 = (1 + F_u^2) du^2 + 2 F_u F_v du dv + (1 + F_v^2) dv^2 \tag{130}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on C $(a \le t \le b)$ can be calculated by the formula:

$$L = \int \int_a^b \sqrt{(1 + F_u^2) u_t^2 + 2 F_u F_v u_t v_t + (1 + F_v)^2 v_t^2} dt \tag{131}$$

1. **Calculation of second quadratic form**
First of all calculate normal unit vector $\mathbf{n}(u, v)$
One can do it in two different ways:
using cross-product:

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times r_v|}, \tag{132}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & F_u \\ 0 & 1 & F_v \end{pmatrix} = -F_u \mathbf{e}_x - F_v \mathbf{e}_y + \mathbf{e}_z$$

Hence according to (132)

$$\mathbf{n} = \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \tag{133}$$

($\mathbf{n}$ is defined up to the sign)

Sometimes it is more easy to calculate $\mathbf{n}$ using that it is proportional to the gradient of equation defining surface:

$$\mathbf{n} \quad \text{is proportional to} \quad \operatorname{grad}(z - F(x, y)) = (-F_x, -F_y, 1) \tag{134}$$

Now for calculating second quadratic form it remains to calculate

$$\mathbf{r}_{uu} = \begin{pmatrix} 0 \\ 0 \\ F_{uu} \end{pmatrix}, \mathbf{r}_{uv} = \begin{pmatrix} 0 \\ 0 \\ F_{uv} \end{pmatrix}, \quad \mathbf{r}_{vv} = \begin{pmatrix} 0 \\ 0 \\ F_{vv} \end{pmatrix}$$

and

$$(\mathbf{r}_{uu}, \mathbf{n}) = \frac{F_{uu}}{\sqrt{1 + F_u^2 + F_v^2 +}},\ (\mathbf{r}_{uv}, \mathbf{n}) = \frac{F_{uv}}{\sqrt{1 + F_u^2 + F_v^2}},\ (\mathbf{r}_{vv}, \mathbf{n}) = \frac{F_{vv}}{\sqrt{1 + F_u^2 + F_v^2}},$$

Hence using expression (133) for unit normal vector $\mathbf{n}$ and (**??**) we come to:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_{uu}, \mathbf{n}) & (\mathbf{r}_{uv}, \mathbf{n}) \\ (\mathbf{r}_{uv}, \mathbf{n}) & (\mathbf{r}_{vv}, \mathbf{n}) \end{pmatrix} = \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} F_{uu} & F_{uv} \\ F_{uv} & F_{vv} \end{pmatrix} \tag{135}$$

Now calculate shape operator:

$$G^{-1} = \frac{1}{1 + F_u^2 + F_v^2} \begin{pmatrix} 1 + F_v^2 & -F_u F_v \\ -F_u F_v & 1 + F_u^2 \end{pmatrix}$$

Hence shape operator:

$$S = G^{-1}A = \frac{1}{(1 + F_u^2 + F_v^2)^{3/2}} \begin{pmatrix} 1 + F_v^2 & -F_u F_v \\ -F_u F_v & 1 + F_u^2 \end{pmatrix} \cdot \begin{pmatrix} F_{uu} & F_{uv} \\ F_{uv} & F_{vv} \end{pmatrix}$$

Calculation of Gaussian and Mean curvatures:

Gaussian curvature according previous considerations (121) is equal to the determinant of the shape operator:

$$K = \det S = \frac{\det A}{\det G} =$$

$$\frac{\frac{F_{uu}F_{vv} - F_{uv}^2}{(1 + F_u^2 + F_v^2)}}{(1 + F_u^2)(1 + F_v^2)^2 - F_u F_v} = \frac{F_{uu}F_{vv} - F_{uv}^2}{(1 + F_u^2 + F_v^2)^2} \tag{136}$$

Mean curvature according to (122) is equal to

$$H = \mathrm{Tr}H = \frac{F_{uu} + F_{vv} + F_v^2 F_{uu} - 2F_u F_v F_{uv} + F_u^2 F_{vv}}{(1 + F_u^2 + F_v^2)^{3/2}} \tag{137}$$

Calculate now Gaussian and mean curvature for cylinder, cone, sphere and saddle. Of course we can use general formulae obtained above. But why not to calculate independently? (It seems to be more interesting and sometimes much easy)

Cylinder

Cylinder is given by the equation $x^2 + y^2 = a^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} \tag{138}$$

Calculation of first quadratic form for cylinder

$$\mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix} \tag{139}$$

,

$$(\mathbf{r}_h, \mathbf{r}_h) = 1, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2$$

and first quadratic form (95) is equal to

$$G = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \tag{140}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}, \quad ds^2 = dh^2 + a^2 d\varphi^2 \tag{141}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cylinder $(a \le t \le b)$ can be calculated by the formula:

$$L = \int_a^b \sqrt{h_t^2 + a^2\varphi_t} dt \tag{142}$$

Calculation of second quadratic form for cylinder

It is evident without any calculations that normal unit vector $\mathbf{n}(u,v)$ to the cylinder (138) is defined by the following formula

$$\mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{143}$$

(unit vector $\mathbf{n}$ is defined up to the sign)

But you can calculate using general formulae above:

Using cross-product:

$$\mathbf{r}_h \times \mathbf{r}_\varphi = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & 1 \\ -a\sin\varphi & a\cos\varphi & 0 \end{pmatrix} = -\cos\varphi \mathbf{e}_x - \sin\varphi \mathbf{e}_y$$

and we come (up to a sign) to the answer (143). (Normal unit vector is defined up to direction $\mathbf{n} \to -\mathbf{n}$)

One can calculate $\mathbf{n}$ using that it is proportional to the gradient of equation defining surface:

$$\mathbf{n} \quad \text{is proportional to} \quad \text{grad}\left(x^2 + y^2 - a^2\right)\right) = (2x, 2y, 0) \tag{144}$$

and come to the same answer (143).

Now calculate second quadratic form:

$$\mathbf{r}_{hh} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{r}_{h\varphi} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -a\cos\varphi \\ -a\sin\varphi \\ 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} (\mathbf{r}_{hh}, \mathbf{n}) & (\mathbf{r}_{h\varphi}, \mathbf{n}) \\ (\mathbf{r}_{h\varphi}, \mathbf{n}) & (\mathbf{r}_{\varphi\varphi}, \mathbf{n}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} \tag{145}$$

Calculation of Shape operator for cylinder:

$$G^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/a^2 \end{pmatrix}$$

Hence shape operator:

$$S = G^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1/a^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1/a \end{pmatrix}$$

Gaussian and Mean curvatures for cylinder:

$$K = \det S = \frac{\det A}{\det G} = 0, H = -\frac{1}{a}$$

Cone

Cone is given by the equation $x^2 + y^2 - kz^2 = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} \tag{146}$$

Calculation of first quadratic form for cone

$$\mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix} \quad \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix} \tag{147}$$

,

$$(\mathbf{r}_h, \mathbf{r}_h) = 1 + k^2, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = k^2h^2$$

and first quadratic form (95) is equal to

$$G = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \tag{148}$$

$$\begin{pmatrix} 1 + k^2 & 0 \\ 0 & k^2h^2 \end{pmatrix}, \quad ds^2 = (1 + k^2)dh^2 + k^2h^2d\varphi^2 \tag{149}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cone ($a \leq t \leq b$) can be calculated by the formula:

$$L = \int_a^b \sqrt{(1 + k^2)h_t^2 + k^2h^2\varphi_t^2}dt \tag{150}$$

Calculation of second quadratic form for cone

In this case easiest way to calculate a unit vector $\mathbf{n}$ is to note that it is proportional to the gradient of equation defining surface:

$$\mathbf{n} \quad \text{is proportional to} \quad \text{grad}\left(x^2 + y^2 - k^2z^2)\right) = (2x, 2y, -2zk^2) \tag{151}$$

Hence

$$\mathbf{n} = \lambda \begin{pmatrix} kh\cos\varphi \\ kh\sin\varphi \\ -k^2h \end{pmatrix}$$

Find λ such that $|\mathbf{n}| = 1$:

$$\mathbf{n} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

Now calculate second quadratic form:

$$\mathbf{r}_{hh} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{r}_{h\varphi} = \begin{pmatrix} -k\sin\varphi \\ k\cos\varphi \\ 0 \end{pmatrix} \quad \mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -kh\cos\varphi \\ -kh\sin\varphi \\ 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} (\mathbf{r}_{hh}, \mathbf{n}) & (\mathbf{r}_{h\varphi}, \mathbf{n}) \\ (\mathbf{r}_{h\varphi}, \mathbf{n}) & (\mathbf{r}_{\varphi\varphi}, \mathbf{n}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{kh}{\sqrt{1+k^2}} \end{pmatrix} \tag{152}$$

Shape operator for cone:

$$G^{-1} = \begin{pmatrix} \frac{1}{1+k^2} & 0 \\ 0 & \frac{1}{k^2h^2} \end{pmatrix}$$

Hence shape operator:

$$S = G^{-1}A = \begin{pmatrix} \frac{1}{1+k^2} & 0 \\ 0 & \frac{1}{k^2h^2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & -\frac{kh}{\sqrt{1+k^2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{kh\sqrt{1+k^2}} \end{pmatrix} =$$

Gaussian and Mean curvatures for cone:

$$K = \det S = \frac{\det A}{\det G} = 0, \qquad H = \frac{1}{kh\sqrt{1+k^2}}$$

Sphere

Sphere is given by the equation $x^2 + y^2 + z^2 = a^2$. Consider the following (standard) parameterisation of this surface:

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases} \tag{153}$$

Calculation of first quadratic form for sphere

$$\mathbf{r}_\theta = \begin{pmatrix} a\cos\theta\cos\varphi \\ a\cos\theta\sin\varphi \\ -a\sin\theta \end{pmatrix} \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\theta\sin\varphi \\ a\sin\theta\cos\varphi \\ 0 \end{pmatrix} \tag{154}$$

,

$$(\mathbf{r}_\theta, \mathbf{r}_\theta) = a^2, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2\sin^2\theta$$

and first quadratic form (95) is equal to

$$G = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \tag{155}$$

$$\begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}, \quad ds^2 = a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2 \tag{156}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$ on the sphere of the radius a ($a \leq t \leq b$) can be calculated by the formula:

$$L = \int_a^b a\sqrt{\theta_t^2 + \sin^2\theta \cdot \varphi_t^2} dt \tag{157}$$

Calculation of second quadratic form for sphere

It is obvious that unit vector $\mathbf{n}$ for the sphere is just

$$\mathbf{n} = \frac{\mathbf{r}}{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Second quadratic form:

$$\mathbf{r}_{\theta\theta} = \begin{pmatrix} -a\sin\theta\cos\varphi \\ -a\sin\theta\sin\varphi \\ -a\cos\theta \end{pmatrix}, \mathbf{r}_{\theta\varphi} = \begin{pmatrix} -a\cos\theta\sin\varphi \\ a\cos\theta\cos\varphi \\ 0 \end{pmatrix} \quad \mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -a\sin\theta\cos\varphi \\ -a\sin\theta\sin\varphi \\ 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} (\mathbf{r}_{\theta\theta}, \mathbf{n}) & (\mathbf{r}_{\theta\varphi}, \mathbf{n}) \\ (\mathbf{r}_{\theta\varphi}, \mathbf{n}) & (\mathbf{r}_{\varphi\varphi}, \mathbf{n}) \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & -a\sin^2\theta \end{pmatrix} \tag{158}$$

Calculation of Shape operator for sphere:

$$G^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2\sin^2\theta} \end{pmatrix}$$

Hence for Shape operator:

$$S = G^{-1}A = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2 \sin^2 \theta} \end{pmatrix} \cdot \begin{pmatrix} -a & 0 \\ 0 & -a \sin^2 \theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{a} & 0 \\ 0 & -\frac{1}{a} \end{pmatrix} =$$

Gaussian and Mean curvatures for Sphere:

$$K = \det S = \frac{\det A}{\det G} = \frac{1}{a^2}, \qquad H = \frac{2}{a}$$

Saddle

Saddle is given by the equation $z - xy = 0$. (This surface contains horizontal and vertical lines...)

Consider the following (standard) parameterisation of this surface:

$$\mathbf{r}(u, v): \quad \begin{cases} x = u \\ y = v \\ z = uv \end{cases} \tag{159}$$

Calculation of first quadratic form for saddle

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix} \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix} \tag{160}$$

,

$$(\mathbf{r}_\theta, \mathbf{r}_\theta) = 1, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2 \sin^2 \theta$$

and first quadratic form (95) is equal to

$$G = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \tag{161}$$

$$\begin{pmatrix} 1 + v^2 & uv \\ uv & 1 + u^2 \end{pmatrix}, \quad ds^2 = (1 + v^2)du^2 + 2uvdudv + (1 + u^2)dv^2 \tag{162}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on the sphere of the radius a $(a \leq t \leq b)$ can be calculated by the formula:

$$L = \int_a^b a\sqrt{(1 + v^2)u_t^2 + 2uvu_tv_t + (1 + u^2)v_t^2 dt} \tag{163}$$

Calculation of second quadratic form for saddle

Calculate $\mathbf{n}$. $\mathrm{grad}(z - xy) = (-y, -x, 1)$. Hence

$$\mathbf{n} = \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}$$

Second quadratic form:

$$\mathbf{r}_{uu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{r}_{uv} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_{vv} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} (\mathbf{r}_{uu}, \mathbf{n}) & (\mathbf{r}_{uv}, \mathbf{n}) \\ (\mathbf{r}_{uv}) & (\mathbf{r}_{vv}, \mathbf{n}) \end{pmatrix} = \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{164}$$

Calculation of Shape operator for saddle:

$$S = G^{-1}A = \frac{1}{(1+u^2+v^2)^{\frac{3}{2}}} \begin{pmatrix} 1+u^2 & -uv \\ -uv & 1+v^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ = & \end{pmatrix}$$

$$\frac{1}{(1+u^2+v^2)^{\frac{3}{2}}} \begin{pmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{pmatrix}$$

Gaussian and Mean curvatures for Saddle:

$$K = \det S = \frac{\det A}{\det G} = -\frac{1}{(1+u^2+v^2)^2}, \quad H = -\frac{2uv}{(1+u^2+v^2)^{\frac{3}{2}}}$$

3 Riemannian manifolds

3.1 Definitions

The Riemannian metric on the manifold M defines the length of the tangent vectors and the length of the curves.

Riemannian metric

Definition

Riemannian metric on n-dimensional manifold M^n defines for every point P the scalar product of tangent vectors in the tangent space T_pM smoothly depending on the point P.

It means that in every coordinate system $(x^1, \dots, x^n)$ a metric G is defined by a matrix $g_{ik}(x)$ such that

- $g_{ik}(x) = g_{ki}(x)$ (Metric is defined by symmetric tensor of second rank)
- $g_{ik}(x)u^iu^k \geq 0$, $g_{ik}(x)u^iu^k = 0$ iff $u^1 = \dots = u^n = 0$ (it is positively defined)
- $g_{ik}(x)$ are smooth functions

For any two vectors

$$\mathbf{A} = \begin{pmatrix} A^1 \\ \cdot \\ \cdot \\ \cdot \\ A^n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} B^1 \\ \cdot \\ \cdot \\ \cdot \\ B^n \end{pmatrix} \tag{165}$$

the scalar product is equal to:

$$G(\mathbf{A}, \mathbf{B}) = A^i G_{ik} B^k = \begin{pmatrix} A^1 \dots A^n \end{pmatrix} \begin{pmatrix} g_{11} & \dots & g_{1n} \\ \dots & \dots & \dots \\ g_{n1} & \dots & g_{nn} \\ & & \end{pmatrix} \begin{pmatrix} B^1 \\ \cdot \\ \cdot \\ \cdot \\ B^n \end{pmatrix} \tag{166}$$

For any two coordinate systems $(x^1, \dots, x^n)$, $(y^1, \dots, y^n)$, $x^i = x^i(y^p)$ matrices $g_{ik}(x)$, $g_{i'k'}(x)$ are related by the relation:

$$\tilde{g}_{pq}(y) = \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \tag{167}$$

It is convenient to write metric:

$$G = g_{ik}(x) dx^i dx^k \tag{168}$$

If $(y^1, \dots, y^n)$ are new coordinates then

$$G = g_{ik}(x) dx^i dx^k = G = dx^i g_{ik}\left(x(y)\right) dx^k =$$

$$\left(dy^p \frac{\partial x^i}{\partial y^p}\right) g_{ik}\left(x(y)\right) \left(\frac{\partial x^k}{\partial y^q}\right) dy^k = dy^p \tilde{g}_{pq}(y) dy^q \tag{169}$$

We come to the formula (167). (We use here condensed notations)

Length of the curve. Let γ: $(x^1(t), \ldots, x^n(t))$ $(a \leq t \leq b)$ be a curve on the Riemannian manifold (M, G). At the every point of the curve the velocity vector (tangent vector) is defined:

$$\mathbf{v}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}^n(t) \end{pmatrix} \tag{170}$$

Then the length of the curve is defined by the integral of the length of velocity vector:

$$L_\gamma = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_a^b \sqrt{g_{ik}(x) \dot{x}^i \dot{x}^k} \tag{171}$$

Bearing in mind that metric (168) defines the length we often write metric in the following form

$$ds^2 = g_{ik} dx^i dx^k \tag{172}$$

For example consider 2-dimensional Riemannian manifold with metric $g_{ik}(u, v)$ $(i, k = 1, 2)$. Then

$$ds^2 = g_{ik} dx^i dx^k = g_{11}(u, v) du^2 + 2g_{12}(u, v) du dv + g_{22}(u, v) dv^2$$

The length of the curve γ: $u = u(t), v = v(t)$, where $t_0 \leq t \leq t_1$ according to (171) is equal to

$$L_\gamma = \int_{t_0}^{t_1} \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_{t_0}^{t_1} \sqrt{g_{ik}(x) \dot{x}^i \dot{x}^k} = \tag{173}$$

$$\int_{t_0}^{t_1} \sqrt{g_{11}\left(u\left(t\right), v\left(t\right)\right) u_t^2 + 2g_{12}\left(u\left(t\right), v\left(t\right)\right) u_t v_t + g_{22}\left(u\left(t\right), v\left(t\right)\right) v_t^2} dt \tag{174}$$

If metric has diagonal form:

$$ds^2 = g_{ik} dx^i dx^k = a(u, v) du^2 + b(u, v) dv^2, \quad (a = g_{11}, b = g_{22})$$

then

$$L_\gamma = \int_{t_0}^{t_1} \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_{t_0}^{t_1} \sqrt{a\left(u\left(t\right), v\left(t\right)\right) u_t^2 + b\left(u\left(t\right), v\left(t\right)\right) v_t^2} dt \qquad (175)$$

First quadratic form is an example of Riemannian metric on the two-dimensional surface. The first quadratic form is defined by the position of surface in three-dimensional ambient space $\mathbf{E}^3$. In the case of Riemannian metric we input it *apriori*.

Consider two-dimensional manifold with metric $(1 + k^2)du^2 + k^2u^2dv^2$.

The length of the curve $u = u(t), v = v(t)$ on this surface is equal to

$$L = \int_{t_0}^{t_1} \sqrt{(1 + k^2)u_t^2 + k^2v_t^2} dt$$

One can compare this metric with metric defined by the first quadratic form on the cone $x^2 + y^2 - k^2z^2 = 0$

Example Consider upper half plane with Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2} \qquad (176)$$

Calculate the length of the vertical line $x = a, 0 < t_0 \leq y \leq t_1$.

3.2 Volume element in Riemannian manifold

The volume element in n-dimensional Riemannian manifold with metric $G = g_{ik}dx^idx^k$ is defined by the formula

$$\sqrt{\det g_{ik}}\, dx^1dx^2 \ldots dx^n \qquad (177)$$

If D is a domain in the n-dimensional Riemannian manifold with metric $G = g_{ik}dx^i$ then its volume is equal to to the integral of volume element over this domain.

$$V(D) = \int_D \sqrt{\det g_{ik}}\, dx^1dx^2 \ldots dx^n \qquad (178)$$

Remark Students who know the concept of exterior forms can read the volume element as

$$\sqrt{\det g_{ik}}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \qquad (179)$$

Note that in the case of $n = 1$ volume is just the length, in the case if $n = 2$ it is area.

Note that the formula (177) gives volume of n-dimensional parallelepiped. In cartesian coordinates we come to standard formula for domain.

Invariance of volume element under changing of coordinates

Prove that volume element is invariant under coordinate transformations, i.e. if $y^1, \dots, y^n$ are new coordinates: $x^1 = x^1(y^1, \dots, y^n), x^2 = x^2(y^1, \dots, y^n)$...,

$$x^i = x^i(y^p), i = 1, \dots, n \quad , p = 1, \dots, n$$

and $\tilde{g}_{pq}(y)$ matrix of the metric in new coordinates:

$$\tilde{g}_{pq}(y) = \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \tag{180}$$

(See formulae (167) and(169)) then

$$\sqrt{\det g_{ik}(x)}\, dx^1 dx^2 \dots dx^n = \sqrt{\det \tilde{g}_{pq}(y)}\, dy^1 dy^2 \dots dy^n \tag{181}$$

This follows from (180). Namely

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)}\, dy^1 dy^2 \dots dy^n$$

Using the fact that $\det(ABC) = \det A \cdot \det B \cdot \det C$ and $\det \left(\frac{\partial x^i}{\partial y^p} \right) = \det \left(\frac{\partial x^k}{\partial y^q} \right)$[6] we see that from the formula above follows:

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)} dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\left(\det \left(\frac{\partial x^i}{\partial y^p} \right) \right)^2} \sqrt{\det g_{ik}(x(y))} dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\det g_{ik}(x(y))} \det \left(\frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \dots dy^n = \tag{182}$$

Now note that

$$\det \left(\frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \dots dy^n = dx^1 \dots dx^n$$

[6]determinant of matrix does not change if we change the matrix on the adjoint, i.e. change columns on rows.

according to the formula for changing coordinates in n-dimensional integral [7]. Hence

$$\sqrt{\det g_{ik}(x(y))}\det\left(\frac{\partial x^i}{\partial y^p}\right)dy^1dy^2\dots dy^n = \sqrt{\det g_{ik}(x(y))}dx^1dx^2\dots dx^n \quad (183)$$

Thus we come to (181).

Example

Consider first very simple example: Volume element of plane in cartesian coordinates, metric $g = dx^2 + dy^2$. Volume element is equal to

$$\sqrt{\det g}dxdy = \sqrt{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}dxdy = dxdy$$

Volume of the domain D is equal to

$$V(D) = \int_D \sqrt{\det g}dxdy = \int_D dxdy$$

If we go to polar coordinates:

$$x = r\cos\varphi, y = r\sin\varphi \quad (184)$$

Then we have for metric:

$$G = dr^2 + r^2d\varphi^2$$

because

$$dx^2 + dy^2 = (dr\cos\varphi - r\sin\varphi d\varphi)^2 + (dr\sin\varphi + r\cos\varphi d\varphi)^2 = dr^2 + r^2d\varphi^2 \quad (185)$$

Volume element in polar coordinates is equal to

$$\sqrt{\det g}drd\varphi = \sqrt{\det\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}drd\varphi = drd\varphi$$

Example. Volume element of the metric of Lobachesvky plane.

[7] Determinant of the matrix $\left(\frac{\partial x^i}{\partial y^p}\right)$ of changing of coordinates is called sometimes Jacobian. Here we consider the case if Jacobian is positive. If Jacobian is negative then formulae above remain valid just the symbol of modulus appears.

In coordinates x, y $(y > 0)$ metric $G = \frac{dx^2+dy^2}{y^2}$, the corresponding matrix $G = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$. Volume element is equal to $\sqrt{\det g} dxdy = \frac{dxdy}{y^2}$

Example Consider the two dimensional plane with Riemannian metrics

$$G = \frac{du^2 + dv^2}{(1 + u^2 + v^2)^2} \tag{186}$$

(It is indeed the sphere in stereographic coordinates)

Calculate its volume element and volume. It is easy to see that:

$$G = \begin{pmatrix} \frac{1}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{1}{(1+u^2+v^2)^2} \end{pmatrix} \qquad \det g = \frac{1}{(1 + u^2 + v^2)^4} \tag{187}$$

and volume element is equal to $\sqrt{\det g} dudv = \frac{dudv}{(1+u^2+v^2)^2}$

The volume (area) of plane will be:

$$\int \frac{dudv}{(1 + u^2 + v^2)^2} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} \frac{du}{(1 + u^2 + v^2)^2} = \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{du}{(1 + u)^{3/2}} = \pi$$

We see that in coordinates (u, v) calculation of the integral is not very easy.

One can consider volume form in polar coordinates $u = r\cos\varphi, v = r\sin\varphi$. Then it is easy to see that according to (185) we have for the metric $G = \frac{du^2+dv^2}{(1+u^2+v^2)^2} = \frac{dr^2+r^2d\varphi^2}{(1+r^2)^2}$ and volume form is equal to $\sqrt{\det g} drd\varphi = \frac{rdrd\varphi}{(1+r^2)^2}$

Now calculation of integral becomes easy:

$$V = \int \frac{rdrd\varphi}{(1 + r^2)^2} = 2\pi \int_0^{\infty} \frac{rdr}{(1 + r^2)^2} = \pi \int_0^{\infty} \frac{du}{(1 + u)^2} = \pi$$

Example Volume element of the segment of the sphere.

Consider sphere of the radius a in Euclidean space with standard Riemanian metric

$$a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2$$

This metric is nothing but first quadratic form on the sphere (see (156)). The volume element is

$$\sqrt{\det g} d\theta d\varphi = \sqrt{\det \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin\theta \end{pmatrix}} d\theta d\varphi = a^2 \sin\theta d\theta d\varphi$$

Now calculate the volume of the segment of the sphere between two parallel planes, i.e. domain restricted by parallels $\theta_1 \leq \theta \leq \theta_0$: Denote by h be the height of this segment. One can see that

$$h = a\cos\theta_0 - a\cos\theta_1 = a(\cos\theta_0 - a\cos\theta_1)$$

There is remarkable formula which express the area of segment via the height h:

$$V = \int_{\theta_1 \leq \theta \leq \theta_0} \left(a^2 \sin\theta\right) d\theta d\varphi = \int_{\theta^0}^{\theta^1} \left(\int_0^{2\pi} \left(a^2 \sin\theta\right) d\varphi\right) d\theta =$$

$$\int_{\theta^1}^{\theta^0} 2\pi a^2 \sin\theta d\theta = 2\pi a^2(\cos\theta_0 - \cos\theta_1) = 2\pi a(a\cos\theta_0 - acos\theta_1) = 2\pi ah \tag{188}$$

E.g. for all the sphere $h = 2a$. We come to $S = 4\pi a^2$. It is remarkable formula: area of the segment is a polynomial function of radius of the sphere and height (Compare with formula for length of the arc of the circle)

3.3 Geodesics

Let A, B are two points on Riemannian manifold (M^n, G). Consider the length of the shortest curve which connects these points More formally consider the set $\mathcal{C}_{AB}$ of the curves which start at the point A and end at the point B. Then the length of the shortest curve (if it exists[8]) is equal to

$$d(A, B) = \inf_{\gamma \in \mathcal{C}_{AB}} L(\gamma) \tag{189}$$

Let $\{u^i(t)\}$ be local coordinates which are defined in the vicinity of the points A and B. If metric is equal to $G = g_{ik} du^i du^k$ in these coordinates then length of arbitrary curve γ: $u^i(t)$ which starts at A and ends at B is equal to

$$L_\gamma = \int_{t_0}^{t_1} \sqrt{g_{ik}(u) \frac{du^i(t)}{dt} \frac{dv^i(t)}{dt}}, \quad u^i(t_0) = u_0^i, u^i(t_1) = u_1^i \tag{190}$$

where u_0^i are coordinates of the initial point A and u_1^i are coordinates of the final point B and the shortest distance is just the inferior of this functional by all the curves beginning at A and ending at B.

[8] we do not consider existence problem and suppose that the shortest curve exist

Example 1

Consider two points in $\mathbf{E}^2$ with cartesian coordinates (x, y) (metric $G = dx^2 + dy^2$): $A = (x_0, y_0)$, $B = (x_1, y_1)$.

Consider an arbitrary curve γ_{AB} $x(t), y(t)$ (such that $x(t_0) = x_0, y(t_0) = t_0$, $x(t_1) = x_1, y(t_1) = t_1$) and consider the line

$$l_{AB} x(t) = x_0 + t(x_1 - x_0), \quad y(t) = y_0 + t(y_1 - y_0) \tag{191}$$

It is easy to see that

$$L_{\gamma_{AB}} = \int_{t_0}^{t_1} \sqrt{x_t^2 + y_t^2} \geq L_{l_{AB}} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

(Technically it is very easy to see in coordinates such that $y_0 = y_1$)

We come to the well-known fact: segment of the line is the shortest distance between two points in Euclidean space.

Generalising the concept of the line as shortest distance between two points for Rieamnnian manifodls we come to *geodesic*

One of definitions of geodesics is following:

Definition The curve γ: $\quad x^i(t)$ is called *geodesic* if of for arbitrary two (enough closed) points $A = x^i(t_0)$, $B = x^i(t_1)$ of this curve the following condition holds: The length of the arc of the curve γ between the points A, B is the shortest, i.e. the length of the arbitrary curve which connects these two points is bigger or equal to the length of this arc.

How to find geodesics?

In general case one have to consider the corresponding variational problem (The Euler-Lagrange equations for functional $\int \sqrt{g_{ik}\dot{x}^i\dot{x}^k}$). In some cases one can come to the answer by elementary methods. For example one can easy to show that geodesics on sphere are great circles. Consider standard Riemannian metrics on the sphere in $\mathbf{E}^3$ with the radius a: Coordinates θ, φ, metrics (first quadratic form):

$$G = a^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{192}$$

Consider two arbitrary points A and B on the sphere. Let (θ_0, φ_0) be coordinates of the point A and (θ_1, φ_1) be coordinates of the point B

Let γ be a curve which connects these points: γ: $\theta(t), \varphi(t)$ such that $\theta(t_0) = \theta_0, \theta(t_1) = \theta_1$, $\varphi(t_0) = \theta_0, \theta(t_1) = \theta_1$ then:

$$L_{\gamma_{AB}} = \int a\sqrt{\theta_t^2 + \sin^2\theta(t)\varphi_t^2}dt \tag{193}$$

Without loss of generality suppose that they have the same latitude, i.e. if (θ_0, φ_0) are coordinates of the point A and (θ_1, φ_1) are coordinates of the point B then $\varphi_0 = \varphi_1$ (if it is not the fact then we can come to this condition rotating the sphere)

Now one can see that the meridian $\varphi = \varphi_0$ is geodesics: Indeed consider an arbitrary curve $\theta(t), \varphi(t)$ which connects the points A, B: $\theta(t_0) = \theta_0, \theta(t_1) = t_1$, $\varphi(t_0) = \varphi(t_1 = \varphi_0$. Compare its length with the length of the meridian which connects the points A, B:

$$\int_{t_0}^{t_1} a\sqrt{\theta_t^2 + \sin^2\theta\varphi_t^2}dt \geq a\int_{t_0}^{t_1} \sqrt{\theta_t^2}dt = a\int_{t_0}^{t_1} \theta_t dt = a(\theta_1 - \theta_0) \qquad (194)$$

the big circles on sphere are geodesics. It corresponds to geometrical intuition: The geodesics on the sphere are the circles of intersection of the sphere with the plane which crosses the centre.

Remark In the integral (194) we considered the smallest arc of the great circle between points A, B.

3.4 Geodesics and isometries

Consider triangles on the Earth.

Let A, B, C be three points on the Earth, e,g, $A = Paris$, $B = Manchester$, $C = Berlin$. Draw the lines connecting these points. They will be arcs of great circles. This triangle is called spheric triangle. The sum of angles of this triangle will not be equal to π. One can prove that if S is a area of spherical triangle then:

$$\text{sum of the angles of spheric triangle} - \pi = \frac{S}{R^2} = KS \qquad (195)$$

where K is Gaussian curvature of the sphere. (In the general case the formula above holds only for small triangles.)

We did not notice this phenomenon in ordinary life because radius of earth is equal to 6400 km.

The fact that sum of the angles is not equal to π is very important property of the sphere. In principal one can guess that Earth is round just drawing big triangles. (See the tale on Aunts in Appendix.)

What happened if surface is cone? with Riemannian metric

$$(1+k)^2du^2 + k^2dv^2$$

The sum of angle of triangle will be again different from ∂ or no.

We know empirically that plane can be bended to the cone. What it means exactly:

Definition The diffeomorphism of Riemannian manifolds which preserve the metrics is called isometry.

In other words isometry is transformation such that preserves distance between the points.

E.g. standard cylinder and standard cone are to the domain in the euclidean plane.

More exactly consider cylinder with punctured line:

$$\mathbf{r}(h,\varphi)\colon\ : \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} ,0<\varphi<2\pi,\ -\infty<h<\infty$$

Then it is isometric to the domain D in the plane (x,y) where $x \in (0,2\pi)$, $-infty < y < \infty$. This isometry has the following form:

$$x = a\varphi, y = h \Rightarrow\ dx^2 + dy^2 = a^2 d\varphi^2 + dh^2 \tag{196}$$

It is easy to see that $a^2d\varphi^2 + dh^2$ is just first quadratic form (Riemannian metric) on the cylinder. The equation (196) corresponds to unfolding of cylinder.

The same with cone: Consider the upper cone with punctured ray:

$$\mathbf{r}(h,\varphi)\colon\ : \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} ,0<\varphi<2\pi,\ 0<h<\infty$$

Its Riemannian metric (first quadratic form) is equal (show it!)

$$ds^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2$$

Establish the isometry with domain in plane e.g. for the case $k = 1$:

$$r = \sqrt{2}h, \quad \Psi = \varphi/\sqrt{2}, \text{ where } \begin{cases} x = r\cos\Psi, \\ y = r\sin\Psi \end{cases}$$

One can see that the map above transforms he metric $dx^2 + dy^2$ into metric on cone. Hence it is isometry. This map corresponds to unfolding of the cone.

Empirically it is evident. One can prove it formally

4 Parallel transport; Gauss–Bonnet Theorem

4.1 Concept of parallel transport

Parallel transport of the vectors is one of the fundamental concept of differential geometry. Here we just give some preliminary ideas and formulate the concept of parallel transport for surfaces embedded in Euclidean space. The detailed approach is founded on the conception of connection and covariant derivative (see the next section).

Let C be a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ and $\gamma(t)$, $t_1 \leq t \leq t_2$ a curve on this surface $\gamma(t)\colon \quad r = \mathbf{r}(t) = \mathbf{r}(u(t), v(t)))$.

Let $\mathbf{X}_1$ be a vector tangent to the surface at the initial point $p = \gamma(t_1)$ of the curve $\gamma(t)$ on the surface: $\mathbf{X}_1 \in T_pC$ $(p = \gamma(t_1))$. We define parallel transport of the vector along the curve:

Definition Let $\gamma(t)$ be a curve on the surface C. Let $\mathbf{X}(t)$ be a family of vectors depending on the parameter t $(t_1 \leq t \leq t_2)$ such that following conditions hold

- For every $t \in [t_1, t_2]$ vector $\mathbf{X}(t)$ is a vector tangent to the surface C attached to the point $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ of the curve $\gamma(t)$.
- $\mathbf{X}(t) = \mathbf{X}_1$ for $t = t_1$
- $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface, i.e.

  $$\frac{d\mathbf{X}(t)}{dt} \text{ is parallel to the normal vector } \mathbf{n}(t), \ \frac{d\mathbf{X}(t)}{dt} = \lambda(t)\mathbf{n}(t) \quad (197)$$

  Recall that normal vector $\mathbf{n}(t)$ is a vector attached to the point $\mathbf{r}(t)$ of the curve $\gamma(t)$ which is orthogonal to the surface C. It can be calculated by the formula:

  $$\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}, \text{ where } \mathbf{N} = [\mathbf{r}_u \times \mathbf{r}_v]$$

  The condition (197) means that only orthogonal component of vector could be changed.

We say that a family $\mathbf{X}(t)$ is a parallel transport of the vector $\mathbf{X}_1$ along a curve $\gamma(t)$ on the surface C. The final vector $\mathbf{X}_2 = \mathbf{X}(t_2)$ is the image of the vector $\mathbf{X}_1$ under the parallel transport along the curve $\gamma(t)$.

Using the relation (197) it is easy to see that the scalar product of two vectors remains invariant under parallel transport. In particularly it means that length of the vector does not change. If $\mathbf{X}(t)$, $\mathbf{Y}(t)$ are parallel transports of vectors $\mathbf{X}_1, \mathbf{Y}_1$ then

$$\frac{d}{dt}\left(\mathbf{X}(t), \mathbf{Y}(t)\right) = \left(\frac{d\mathbf{X}(t)}{dt}, \mathbf{Y}(t)\right) + \left(\mathbf{X}(t), \frac{d\mathbf{Y}(t)}{dt}\right) = 0$$

because vector $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the vector $\mathbf{Y}(t)$ and vector $\frac{d\mathbf{Y}(t)}{dt}$ is orthogonal to the vector $\mathbf{X}(t)$. In particularly length does not change:

$$\frac{d}{dt}|\mathbf{X}(t)|^2 = \frac{d}{dt}(\mathbf{X}(t), \mathbf{X}(t)) = 2(\frac{d\mathbf{X}(t)}{dt}, \mathbf{X}(t)) = 2(\lambda(t)\mathbf{n}(t), \mathbf{X}(t)) = 0 \tag{198}$$

Remark The relation (197) shows how the surface is engaged in the parallel transport. Note that it is non-sense to put the right hand side of the equation (197) equal to zero: In general a tangent vector ceased to be tangent to the surface if it is not changed! (E.g. consider the vector which transports along the great circle on the sphere)

4.2 Parallel transport of vectors tangent to the sphere.

1. In the case if surface is a plane then everything is easy. If vector $\mathbf{X}_1$ is tangent to the plane at the given point, it is tangent at all the points. Vector does not change under parallel transport $\mathbf{X}(t) \equiv \mathbf{X}$.

Consider a case of parallel transport along curves on the sphere.

Consider on the sphere tangent vectors:

$$\mathbf{r}_\theta = \begin{pmatrix} a\cos\theta\cos\varphi \\ a\cos\theta\sin\varphi \\ -a\sin\theta \end{pmatrix} \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\theta\sin\varphi \\ a\sin\theta\cos\varphi \\ 0 \end{pmatrix} \tag{199}$$

attached at the point $\mathbf{r}(\theta, \varphi) = \begin{pmatrix} a\sin\theta\cos\varphi \\ a\sin\theta\sin\varphi \\ a\cos\theta \end{pmatrix}$. One can see that

$$(\mathbf{r}_\theta, \mathbf{r}_\theta) = a, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2\sin^2\theta$$

It is convenient to introduce vectors which are parallel to these vectors but have unit length:

$$\mathbf{e}_\theta = \frac{\mathbf{r}_\theta}{a}, \quad \mathbf{e}_\varphi = \frac{\mathbf{r}_\varphi}{a\sin\theta} \qquad (\mathbf{e}_\theta, \mathbf{e}_\theta) = 1, (\mathbf{e}_\theta, \mathbf{e}_\varphi) = 0, (\mathbf{e}_\varphi, \mathbf{e}_\varphi) = 1\,. \tag{200}$$

How these vectors change if we move along parallel (i.e. what is the value of $\frac{\partial \mathbf{e}_\theta}{\partial\varphi}, \frac{\partial \mathbf{e}_\varphi}{\partial\varphi}$); how these vectors change if we move along meridians (i.e. what is the value of $\frac{\partial \mathbf{e}_\theta}{\partial\theta}, \frac{\partial \mathbf{e}_\varphi}{\partial\theta}$). First of all recall that unit normal vector to the sphere at the point θ, φ is equal to $\frac{\mathbf{r}(\theta,\varphi)}{a}$:

$$\mathbf{n}(\theta,\varphi) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Now calculate:

$$\frac{\partial \mathbf{e}_\theta}{\partial\theta} = \frac{\partial}{\partial\theta}\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ -\cos\theta \end{pmatrix} = -\mathbf{n} \tag{201}$$

,

$$\frac{\partial \mathbf{e}_\theta}{\partial\varphi} = \frac{\partial}{\partial\varphi}\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} = \cos\theta\mathbf{e}_\varphi, \tag{202}$$

,

$$\frac{\partial \mathbf{e}_\varphi}{\partial\theta} = \frac{\partial}{\partial\theta}\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} = 0, \tag{203}$$

$$\frac{\partial \mathbf{e}_\varphi}{\partial\varphi} = \frac{\partial}{\partial\varphi}\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} = -\sin\theta\mathbf{n} - \cos\theta\mathbf{e}_\theta, \tag{204}$$

Some of these formulaes are intuitively evident: For example formula (201) which means that family of the vectors $\mathbf{e}_\theta(\theta)$ is just parallel transport along meridian, because its derivation is equal to $-\mathbf{n}$.

Another intuitively evident example: consider the meridian $\theta(t) = t$, $\varphi(t) = \varphi_0$, $0 \le t \le \pi$. It is easy to see that the vector field

$$\mathbf{X}(t) = \mathbf{e}_\theta(\theta(t), \varphi_0) = \begin{pmatrix} \cos\theta(t)\cos\varphi_0 \\ \cos\theta(t)\sin\varphi_0 \\ -\sin\theta(t) \end{pmatrix}$$

attached at the point $(\theta(t), \varphi_0)$ is a parallel transport because for family of vectors $\mathbf{X}(t)$ all the conditions of parallel transport are satisfied. In particular according to (201)

$$\frac{d\mathbf{X}(t)}{dt} = \frac{d\theta(t)}{dt}\frac{\partial}{\partial\theta}\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix} = -\mathbf{n}(\theta(t), \varphi_0)$$

Now consider an example which is intuitively not-evident.

Example. Calculate parallel transport of the vector $\mathbf{e}_\varphi$ along the parallel. On the sphere of the radius a consider the parallel

$$\theta(t) = \theta_0, \varphi(t) = t, \quad 0 \leq t \leq 2\pi \tag{205}$$

In cartesian coordinates equation of parallel will be:

$$\mathbf{r}(t) = \begin{pmatrix}a\sin\theta(t)\cos\varphi(t)\\ a\sin\theta(t)\sin\varphi(t)\\ -a\cos\theta(t)\end{pmatrix} = \begin{pmatrix}a\sin\theta_0\cos t\\ a\sin\theta(t)\sin t\\ -a\cos\theta_0\end{pmatrix}, \quad 0 \leq t \leq 2\pi \tag{206}$$

It is easy to see that the family of the vectors $\mathbf{e}_\varphi(\theta_0, \varphi(t))$ on parallel, is not parallel transport! because $\frac{d\mathbf{e}_\varphi(\theta_0,\varphi(t))}{dt} = \frac{d\mathbf{e}_\varphi(\theta_0,\varphi)}{d\varphi}$ is not equal to zero (see (204) above). Let a family of vectors $\mathbf{X}(t)$ be a parallel transport of the vector $\mathbf{e}_\varphi$ along the parallel (205): $\mathbf{X}(t) = a(t)\mathbf{e}_\theta(t) + b(t)\mathbf{e}_\varphi(t)$ where $a(t), b(t)$ are components of the tangent vector $\mathbf{X}(t)$ with respect to the basis $\mathbf{e}_\theta, \mathbf{e}_\varphi$ at the point $\theta = \theta_0, \varphi = t$ on the sphere. Initial conditions for coefficients are $a(t)|_{t=0} = 0, b(t)|_{t=0} = 1$ According to the definition of parallel transport and formulae (201)—(204) we have:

$$\frac{d\mathbf{X}(t)}{dt} = \frac{d\left(a(t)\mathbf{e}_\theta(t) + b(t)\mathbf{e}_\varphi(t)\right)}{dt} = \left(\frac{da(t)}{dt}\right)\mathbf{e}_\theta + a(t)\cos\theta_0\mathbf{e}_\varphi + \frac{db(t)}{dt}\mathbf{e}_\varphi +$$

$$b(t)\left(-\sin\theta_0\mathbf{n} - \cos\theta\mathbf{e}_\theta\right) =$$

$$= \left(\frac{da(t)}{dt} - b(t)\cos\theta_0\right)\mathbf{e}_\theta + \left(\frac{db(t)}{dt} + a(t)\cos\theta_0\right)\mathbf{e}_\varphi - b(t)\sin\theta_0\mathbf{n} \tag{207}$$

Under parallel transport only orthogonal component of the vector changes. Hence we come to differential equations

$$\begin{cases}\frac{da(t)}{dt} - wb(t) = 0\\ \frac{db(t)}{dt} + wa(t)\end{cases} \qquad a(0) = 0, b(0) = 0, w = \cos\theta_0 \tag{208}$$

The solution of these equations is $a(t) = \sin wt, b(t) = \cos wt$. We come to the following answer: parallel transport along parallel $\theta = \theta_0$ of the initial vector $\mathbf{e}_\varphi$ is the family

$$\mathbf{X}(t) = \sin wt\, \mathbf{e}_\theta + \cos wt\, \mathbf{e}_\varphi, w = \cos\theta_0 \tag{209}$$

During traveling along the parallel $\theta = \theta_0$ the $\mathbf{e}_\theta$ component becomes non-zero At the end of the traveling the initial vector $\mathbf{X}(t)|_{t=0} = \mathbf{e}_\varphi$ becomes $\mathbf{X}(t)|_{t=2\pi} = \sin 2\pi w \mathbf{e}_\theta + \cos 2\pi w \mathbf{e}_\varphi$: **the vector $\mathbf{e}_\varphi$ after woldtrip traveling along the parallel $\theta = \theta_0$ transforms to the vector $\sin(2\pi\cos\theta_0)\mathbf{e}_\theta + \cos(2\pi\cos\theta_0)\mathbf{e}_\varphi$. In particularly this means that the vector $\mathbf{e}_\varphi$ after parallel transport will rotate on the angle**

$$\text{angle of rotation} = 2\pi\cos\theta_0$$

Compare the angle of rotation with the area of the segment of the sphere above the parallel $\theta = \theta_0$. According to the formula (188) area of this segment is equal to $S = 2\pi a h = 2\pi a^2(1 - \cos\theta_0)$. On the other hand Gaussian curvature of the sphere is equal to $\frac{1}{a^2}$. Hence we see that up to the sign angle of rotation is equal to area of the seqment divided on the Gaussian curvature:

$$\Delta\varphi = \pm\frac{S}{K} = \pm 2\pi\cos\theta_0 \tag{210}$$

4.3 Parallel transport along a closed curve on arbitrary surface.

The formula above for the parallel transport along parallel on the sphere keeps in the general case.

Theorem Let M be a surface in $\mathbf{E}^3$. Let $\gamma(t)\colon \mathbf{r}(t), t_1 \le t \le t_2, \mathbf{r}(t_1) = \mathbf{r}(t_2)$ be a closed curve on the surface M such that it is a boundary of domain D of the surface M. (We suppose that the domain D is bounded an orientate.) Let $\mathbf{X}(t)$ be a parallel transport of the arbitrary tangent vector along this closed curve, i.e. for every $t \in [t_1, t_2]$ $\mathbf{X}(t)$ is a vector tangent to the surface M attached at the point $\mathbf{r}(t)$ of the curve $\gamma(t)$ such that $\frac{d\mathbf{X}(t)}{dt}$ is the vector orthogonal to the surface. Consider initial and final vectors $\mathbf{X}(t_1), \mathbf{X}(t_2)$. They have the same length. The angle $\Delta\varphi$ between these vectors is equal to the integral of Gaussian curvature over the domain D:

$$\delta\varphi = \pm\int_D K\sqrt{\det g}dudv \tag{211}$$

The calculations above for traveling along the parallel are just example of this Theorem. The integral of Gaussian curvature over the domain above parallel $\theta = \theta_0$ is equal to $K \cdot 2\pi a(1-\cos\theta_0) = \frac{1}{a^2} \cdot 2\pi a^2(1-\cos\theta_0) = 2\pi(1-\cos\theta_0)$. This is equal to the angle of rotation $2\pi\cos\theta_0$ (up to a sign and modulo 2π). Another simple

Example. Consider on the sphere $x^2+y^2+z^2=a^2$ points $A=(0,0,1)$, $B=(1,0,0)$ and $C=(0,1,0)$. Consider arcs of great circles which connect these points. Consider the vector $\mathbf{e}_x$ attached at the point A. This vector is tangent to the sphere. It is easy to see that under parallel transport along the arc AB it will transform at the point B to the vector $-\mathbf{e}_z$. The vector $-\mathbf{e}_z$ under parallel transport along the arc BC will remain the same vector $-\mathbf{e}_z$. And finally under parallel transport along the arc CA the vector $-\mathbf{e}_z$ will transform at the point A to the vector $-\mathbf{e}_y$. We see that under traveling along the curvilinear triangle ABC vector $\mathbf{e}_x$ becomes the vector $-\mathbf{e}_y$, i.e. it rotates on the angle $\frac{\pi}{2}$. It is just the integral of the curvature $\frac{1}{a^2}$ over the triangle ABC: $K \cdot S = \frac{1}{a^2} \cdot \frac{4\pi a^2}{8} = \frac{\pi}{2}$.

4.4 Gauss Bonnet Theorem

Consider the integral of curvature over whole closed surface M. According to the Theorem above the answer has to be equal to 0 (modulo 2π), i.e. $2\pi N$ where N is an integer, because this integral is a limit when we consider very small curve. We come to the formula:

$$\int_D K dv = \pm \int_D K\sqrt{\det g}\, du dv = 2\pi N$$

What is the value of integer N?

We present now one remarkable Theorem which answers this question. For more detail see the section 6.

Let M be a closed orientable surface.[9] All these surfaces can be classified up to a diffeomorphism. Namely arbitrary closed oriented surface M is diffeomorphic either to sphere (zero holes), or torus (one hole), or pretzel (two holes),... "Number k" of holes is intuitively evident characteristic of the

[9]Closed means compact surface without boundaries. Intuitively orientability means that one can define out and inner side of the surface. In terms of normal vectors orientability means that one can define the continuous field of normal vectors at all the points of M. The direction of normal vectors at any point defines outward direction. Orientable surface is called oriented if the direction of normal vector is chosen.

surface. It is related with very important characteristic—Euler characteristic $\kappa(M)$ by the following formula:

$$\kappa(M) = 2(1 - g(M)), \quad \text{where } g \text{ is number of holes} \tag{212}$$

Remark What we have called here "holes" in a surface is often referred to as "handles" attached o the sphere, so that the sphere itself does not have any handles, the torus has one handle, the pretzel has two handles and so on. The number of handles is also called genus.

Euler characteristic appears in many different way. The simplest appearance is the following:

Consider on the surface M an arbitrary set of points (vertices) connected with edges (graph on the surface) such that surface is divided on polygons with (curvilinear sides)—plaquets. ("Map of world")

Denote by P number of plaquets (countries of the map)

Denote by E number of edges (boundaries between countries)

Denote by V number of vertices.

Then it turns out that

$$P - E + V = \kappa(M) \tag{213}$$

It does not depend on the graph, it depends only on how much holes has surface.

E.g. for every graph on M, $P - E + V = 2$ if M is diffeomorphic to sphere. For every graph on M $P-E+V = 0$ if M is diffeomorphic to torus.

Now we formulate Gauß-Bonnet Theorem.

Let M be closed oriented surface in $\mathbf{E}^3$.

Let $g = g_{ik}du^i du^k d$ be induced Riemanian metric on this surface, i.e. first quadratic form and $K(p)$ Gaussian curvature at any point p of this surface.

Recall that sign of Gaussian curvature does not depend on the orientation. If we change direction of normal vector $\mathbf{n} \to -\mathbf{n}$ then both principal curvatures change the sign and Gaussian curvature $K = \det A/\det G$ does not change the sign [10].

[10] For an arbitrary point p of the surface M one can always choose cartesian coordinates (x, y, z) such that surface in a vicinity of this spoint is defined by the equation $z = ax^2 + bx^2 + \dots$, where dots means terms of the order higher than 2. Then Gaussian curvature at this point will be equal to ab. If a, b have the same sign then a surfaces looks as paraboloid in the vicinity of the point p. If If a, b have different signs then a surfaces looks as saddle in the vicinity of the point p. Gaussian curvature is positive if $ab > 0$ (case of paraboloid) and negative if $ab < 0$ saddle

Theorem (Gauß -Bonnet) The integral of Gaussian curvature over the closed compact oriented surface M is equal to 2π multiplied by the Euler characteristic of the surface M

$$\frac{1}{2\pi}\int_M K\sqrt{\det g}\,dudv = \kappa(M) = 2(1 - \text{number of holes}) \tag{214}$$

In particular for the surface M diffeomorphic to the sphere $\kappa(M) = 2$, for the surface diffeomorphic to the torus it is equal to 0.

The value of the integral does not change under continuous deformations of surface! It is integer number (up to the factor π) which characterises topology of the surface.

E.g. consider surface M which is diffeomorphic to the sphere. If it is sphere of the radius R then curvature is equal to $\frac{1}{R^2}$, area of the sphere is equal to $4\pi R^2$ and left hand side is equal to $\frac{4\pi}{2\pi} = 2$.

If surface M is an arbitrary surface diffeomorphic to M then metrics and curvature depend from point to the point, Gauß -Bonnet states that integral nevertheless remains unchanged.

Very simple but impressive corollary:

Let M be surface diffeomorphic to sphere in $\mathbf{E}^3$. Then there exists at least one point where Gaussian curvature is positive.

Proof: Suppose it is not right. Then $\int_M K\sqrt{\det g}dudv \leq 0$. On the other hand according to the Theorem it is equal to 4π. Contradiction.

In the first section in the subsection "Integrals of curvature along the plane curve" we proved that the integral of curvature over closed convex curve is equal to 2π. This Theorem seems to be "ancestor" of Gauß-Bonnet Theorem[11].

5 Levi Civita Connection on Riemannian manifold

5.1 Affine connection

How to differentiate functions, vector fields on a (smooth)manifold M? along vector fields.

[11] Note that there is a following deep difference: Gaussian curvature is internal property of the surface: it does not depend on isometries of surface. Curvature of curve depends on the position of the curve in ambient space.

First of all consider differentiation of functions along vector fields.

Let $\mathbf{X} = \mathbf{X^i}(\mathbf{x})\mathbf{e_i}(\mathbf{x})$ be a vector field on M ($\mathbf{e}_i(x) = \frac{\partial}{\partial x^i}$). Recall that vector field [12] $\mathbf{X} = \mathbf{X^i}\mathbf{e_i}$ defines at the every point x_0 an infinitezimal curve: $x^i(t) = x_0^i + tX^i$.

Let f be an arbitrary (smooth) function on M and $\mathbf{X} = X^i \frac{\partial}{\partial x^i}$. Then derivative of function f along vector field $\mathbf{X} = X^i \frac{\partial}{\partial x^i}$ is equal to

$$\nabla_{\mathbf{X}} f = X^i \frac{\partial f}{\partial x^i}$$

The geometrical meaning of this definition is following: If $\mathbf{X}$ is a velocity vector of the curve $x^i(t)$ at the point $x_0^i = x^i(t)$ at the "time" $t = 0$ then the value of the derivative $\nabla_{\mathbf{X}} f$ at the point $x_0^i = x^i(0)$ is equal just to the derivative by t of the function $f(x^i(t))$ at the "time" $t = 0$:

$$\text{if} \quad X^i(x)\big|_{x_0 = x(0)} = \frac{dx^i(t)}{dt}\big|_{t=0}, \quad \text{then} \quad \nabla_{\mathbf{X}} f\big|_{x^i = x^i(0)} = \frac{d}{dt} f\left(x^i\left(t\right)\right)\big|_{t=0} \tag{215}$$

One can see that the operation ∇_X satisfies the following conditions:

- $\nabla_{\mathbf{X}}\left(af + bg\right) = a\nabla_{\mathbf{X}} f + b\nabla_{\mathbf{X}} g$ where $\lambda \in \mathbf{R}$ (linearity over numbers)
- $\nabla_{h\mathbf{X}+g\mathbf{Y}}(f) = h\nabla_{\mathbf{X}}(f) + g\nabla_{\mathbf{Y}}(f)$ (linearity over functions)
- $\nabla_{\mathbf{X}}(\lambda fg) = f\nabla_{\mathbf{X}}(\lambda g) + g\nabla_{\mathbf{X}}(\lambda f)$ (Leibnitz rule)

(216)

How to define differentiation of vector fields along vector fields.

The formula (215) cannot be generalized, because vectors at the point x_0 and $x_0 + tX$ are vectors from different vector spaces. (We cannot substract the vector from one vector space from the vector from the another vector space, because *apriori* we cannot compare vectors from different vector space)

One have to define an operation of transport of vectors from the space $T_{x_0}M$ to the point $T_{x_0+tX}M$[13].

Try to define the operation ∇ on vector fields such that conditions (216) above be satisfied.

Definition Affine connection on M is the *operation* ∇ which assigns to every vector field $\mathbf{X}$ a linear map, (but not $C(M)$-linear map!) (i.e. a map which is linear over numbers not over functions) $\nabla_{\mathbf{X}}$ on the space $\mathcal{O}(M)$ of vector fields:

$$\nabla_{\mathbf{X}}\left(a\mathbf{Y} + b\mathbf{Z}\right) = a\nabla_{\mathbf{X}}\mathbf{Y} + b\nabla_{\mathbf{X}}\mathbf{Z}, \qquad \text{for every constants } a, b \tag{217}$$

(Compare the first condition in (216)).
which satisfies the following conditions:

[12] here like always we suppose by default the summation over repeated indices. E.g.$\mathbf{X} = X^i\mathbf{e}_i$ is nothing but $\mathbf{X} = \sum_{i=1}^n X^i\mathbf{e}_i$

[13] one can define this transport depending on the path from the point $T_{x_0}M$ to the point $T_{x_0+tX}M$

- for arbitrary functions f, g on M

$$\nabla_{f\mathbf{X}+\mathbf{g}\mathbf{Y}}(\mathbf{Z}) = f\nabla_{\mathbf{X}}(\mathbf{Z}) + g\nabla_{\mathbf{Y}}(\mathbf{Z}) \qquad (C(M)\text{-linearity}) \tag{218}$$

(compare with second condition in (216))

- for arbitrary function f

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\nabla_{\mathbf{X}} f)\,\mathbf{Y} + f\nabla_{\mathbf{X}}(\mathbf{Y}) \qquad \text{(Leibnitz rule)} \tag{219}$$

(Compare with Leibnitz rule in (216)).

The vector field $\nabla_{\mathbf{X}}\mathbf{Y}$ is called covariant derivative of vector field $\mathbf{Y}$ along the vector field $\mathbf{X}$.

Write down explicit formulae.

Using properties above one can see that

$$\nabla_{\mathbf{X}}\mathbf{Y} = \nabla_{X^i\mathbf{e}_i} Y^k\mathbf{e}_k = X^i\left(\nabla_i\left(Y^k\mathbf{e}_k\right)\right), \qquad \text{where } \nabla_i = \nabla_{\mathbf{e}_i} \tag{220}$$

Then according to (218)

$$\nabla_i\left(Y^k\mathbf{e}_k\right) = \nabla_i\left(Y^k\right)\mathbf{e}_k + Y^k\nabla_i\mathbf{e}_k$$

The vector field $\nabla_i\mathbf{e}_k$ can be decomposed by basis:

$$\nabla_i\mathbf{e}_k = \Gamma^m_{ik}\mathbf{e}_m$$

and

$$\nabla_i\left(Y^k\mathbf{e}_k\right) = \frac{\partial Y^k(x)}{\partial x^i}\mathbf{e}_k + Y^k\Gamma^m_{ik}\mathbf{e}_m, \tag{221}$$

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^i\frac{\partial Y^m(x)}{\partial x^i}\mathbf{e}_m + X^iY^k\Gamma^m_{ik}\mathbf{e}_m, \tag{222}$$

In components

$$(\nabla_{\mathbf{X}}\mathbf{Y})^m = X^i\left(\frac{\partial Y^m(x)}{\partial x^i} + Y^k\Gamma^m_{ik}\right)$$

Coefficients $\{\Gamma^m_{ik}\}$ are called *Christoffel symbols.*

Example of affine connection

It follows from the properties of connection that it is suffice to define connection at vector fields which form basis at the every point.

For example consider n-dimensional euclidean space $\mathbf{E}^n$. Consider n vector fields $\mathbf{r}_1(x), \ldots, \mathbf{r}_n(x)$ such that

$$\text{they are linearly independent at any point of } \mathbf{E}^n \tag{223}$$

In other words they form basis at every point.

Define connection such that it obeys the condition:

$$\nabla_{\mathbf{r}_k(x)}\mathbf{r}_q(x) = 0 \text{ for every } k, q = 1, 2, 3, \ldots n \tag{224}$$

In the case if $\{\mathbf{r}_k(x)\}$ is just the standard basis on $\mathbf{E}^n$ then we come to the standard connection. We consider *arbitrary vector fields* satisfying the condition (223). In the case if $\{\mathbf{r}_k(x)\}$ is just the standard basis on $\mathbf{E}^n$ then we come to the standard connection in $\mathbf{E}^n$.

The relation (224) defines covariant derivative $\nabla_{\mathbf{X}}\mathbf{Y}$ for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$.

Indeed expand vector fields with respect to the basis $\{\mathbf{r}_k(x)\}$:

$$\mathbf{X} = X^k(x)\mathbf{r}_k(x), \quad \mathbf{Y} = \mathbf{X} = X^q(x)\mathbf{r}_q(x)$$

(here as always in condensed notations we have summation over indices k, q).

Using relations (217)—(219) we see that in the same way as in (221)

$$\nabla_{\mathbf{X}}\mathbf{Y} = \nabla_{X^k(x)\mathbf{r}_k}\left(Y^q(x)\mathbf{r}_q\right) = \tag{225}$$

$$\left[X^k(x)\underbrace{\left(\nabla_{\mathbf{r}_k}Y^q(x)\right)}_{I} + X^k(x)\underbrace{\left(\nabla_{\mathbf{r}_k}\mathbf{r}_k\right)}_{II}Y^q\right]\mathbf{r}_q$$

Second term vanishes according (224). The first term is just derivative of function along vector field (see (215)).

Remark Of course we can define the connection on the basis taking the right hand side in (224) not zero, but arbitrary $\Gamma^i km$. In this case second term in the last relation will not vanish.

5.2 Parallel transport II

Let M be a manifold with affine connection ∇ on it.

Let γ: $x^i(t)$, $a \le t \le b$ be a curve on M Let $\mathbf{v}(t) = \frac{dx(t)}{dt}$ be a velocity vector of the curve γ. Let $\mathbf{Y}(x)$ be an arbitrary vector field on M. Then one can consider the vector field covariant derivative $\nabla_{\mathbf{v}}\mathbf{Y}$ defined just at the points $x(t)$ of the curve γ:

$$\nabla_{\mathbf{v}}\mathbf{Y}\big|_t = v^i(t)\nabla_i(Y^m\mathbf{e}_m)|_t = \left(v^i(t)\frac{\partial Y^m(x)}{\partial x^i}|_{x(t)} + \Gamma^m_{ik}Y^k|_{x(t)}\right)\mathbf{e}_m \tag{226}$$

Note that vector field $\nabla_{\mathbf{v}}\mathbf{Y}$ is well-defined at the points of the curve even if the vector field $\mathbf{Y}$ is defined only at the points of this curve, because

$$v^i(t)\frac{\partial Y^m(x)}{\partial x^i}|_{x(t)} = \frac{dY^m(x(t))}{dt}|_{x(t)}$$

Hence for every vector field $\mathbf{Y}(t)$ attached at the points $x(t)$ of the curve ($\mathbf{Y}(t) \in T_{x(t)}M$) the connection ∇ defines at the points of this curve a vector field $\nabla_{\mathbf{v}}\mathbf{Y}$:

$$\nabla_{\mathbf{v}}\mathbf{Y}\big|_t = \frac{dY^m(x(t))}{dt}|_{x(t)} + v^i(t)\Gamma^m_{ik}Y^k|_{x(t)} \tag{227}$$

It is covariant derivative of $\mathbf{Y}$ along the curve.

Now we are able to define parallel transport of vector field.

Definition

Let $\gamma\colon x^i(t)$, $a \le t \le b$ be a curve on M.

The family of vectors $\mathbf{Y}(t)$, where the vector $\mathbf{Y}(t)$ is a vector attached at the point $x(t)$ ($a \le t \le b$, $\mathbf{Y}(t) \in T_{x(t)}M$) is called a *parallel transport* of the initial vector $\mathbf{Y}(t_0)$ along a curve $\gamma\colon\ x^i(t)$ if covariant derivative $\nabla_{\mathbf{v}}\mathbf{Y} \equiv 0$ at all the points of curve, i.e.

$$\frac{dY^i(x(t))}{dt} + v^k(x(t))Y^m(x(t))\Gamma^i_{km}(x(t)) = 0, \quad \text{where} \quad v^i(t) = \frac{dx^i(t)}{dt} \tag{228}$$

In the case if $\Gamma^i_{km}(x(t)) = 0$ then parallel transport of the vector means just preserving the components of the vector. We consider later really interesting examples of parallel transport, when the final vector depends on the curve, i.e. if γ_1, γ_2 are two curves connecting the points A, B of the manifold, then parallel transport of initial vector $\mathbf{Y}$ from the point A to the point B is different for curves γ_1, γ_2.

5.3 Levi-Civita connection

Let M be a Riemannian manifold with metrics G.

Recall that metrics defines scalar product of vector fields:

$$<\mathbf{X}, \mathbf{Y}> = g_{ik}X^iY^k$$

Let ∇ be a connection on M:

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^i\frac{\partial Y^k}{\partial x^i} + X^iY^k\Gamma^m_{ik}$$

We say that this connection is *symmetric connection* if Christophel symbol Γ^m_{ik} satisfies the condition [14]:

$$\Gamma^m_{ik} = \Gamma^m_{ki}$$

Definition Symmetric connection ∇ on the Riemanian manifold (M, G) is called Levi-Civita connection if it preserves the scalar product It means the following:

Let $\gamma\colon x(t)$ be arbitrary curve on M and $\mathbf{X}, \mathbf{Y}$ arbitrary vectors attached at the initial point of this curve. Let $\mathbf{X}(t), \mathbf{Y}(t)$ be parallel transport of these vectors. Then scalar product $(\mathbf{X}(t), \mathbf{Y}(t))$ of these vectors does not depend on t. In particular it means that under parallel transport length of the vector does not change: (consider $\mathbf{X} = \mathbf{Y}$ we see that $(\mathbf{X}(t), \mathbf{Y}(t)) = |\mathbf{X}|^2$ is preserved) and the angle between vectors does not change: $((\mathbf{X}(t), \mathbf{Y}(t)) = |\mathbf{X}||\mathbf{Y}|\cos\varphi)$.

[14] In a more invariant way one can define define a symmetric connection ∇ as a connection which satisfies the condition:

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0$$

The left hand side of the formula above defines the torsion of the connection.

One can show that Levi-Civita connection on Riemanian manifold can be uniquely defined by the Riemanian metric $G = G_{ik}dx^i dx^k$ (See the subsection in Appendix)

Here we consider only the special case of two-dimensional surface, in $\mathbf{E}^3$ when Riemanian metric is defined by first quadratic form, i.e. it is induced by the metric on $\mathbf{E}^3$.

In this case vectors tangent to surface can be viewed as vectors in $\mathbf{E}^3$ and their length is just the standard length of the vector in $\mathbf{E}^3$.

The Levi-Civita connection defines the parallel transport of an arbitrary vector along an arbitrary curve.

Using the fact that connection have to be symmetric one can prove the following Proposition:

Proposition

Consider surface M in $\mathbf{E}^3$ with induced Riemanian metric, i.e. with metric defined by the first quadratic form.

Let γ: $\mathbf{r}(t), a \leq t \leq b$ be arbitrary curve on this surface. Let $\mathbf{Y}(t)$ be parallel transport of the vector $\mathbf{Y}$ along this curve with respect to Levi-Civita connection on the surface M. Then for vector field $\mathbf{Y}(t)$ the following conditions hold:

- $\mathbf{Y}(t)|_{t=a} = \mathbf{Y}$ *(initial condition)*
- $\mathbf{Y}(t)$ *is always tangent to the surface*

$$(\mathbf{Y}(t), \mathbf{n}(t)) = 0, \quad \text{where } \mathbf{n}(t) \text{ is normal vector} \tag{229}$$

- *only normal component of $\mathbf{Y}$ changes, i.e. derivative $\frac{d\mathbf{Y}(t)}{dt}$ is proportional to the normal vector:*

$$\frac{d\mathbf{Y}(t)}{dt} = \lambda(t)\mathbf{n}(t) \tag{230}$$

These conditions uniquely define parallel transport[15].

In particular it follows from these conditions that the *length of the vector $\mathbf{Y}(t)$ is preserved*:

$$\frac{d}{dt}|\mathbf{Y}(t)|^2 = \frac{d}{dt}(\mathbf{Y}(t), \mathbf{Y}(t)) = 2\left(\frac{d\mathbf{Y}(t)}{dt}, \mathbf{Y}(t)\right) = (\lambda(t)\mathbf{n}(t), \mathbf{Y}(t)) = 0$$

The statement of this Proposition is very useful criterium for constructing parallel transport of vector fields along curves in surfaces. Consider

Example Consider a sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ with induced Riemanian metric=first quadratic form. Consider the vector $\mathbf{Y} = \mathbf{e}_y + \mathbf{e}_z$ attached at the point $A = (1, 0, 0)$. It is evident that this vector is tangent to the sphere at the point A. Consider the arc of the great circle $x = \cos t, y = \sin t, z = 0$, $0 \leq t \leq \pi/2$ beginning at the point A. Find parallel transport of the vector $\mathbf{Y}$ along this curve. In this simple case it is very easy to guess an answer, and then to check is it right or no. Vector $\mathbf{e}_z$ at all the points of the circle $x = \cos t, y = \sin t, z = 0$ remains tangent vector. Vector $\mathbf{e}_y$ has to be transformed to remain tangent. (it has to be rotated: $\mathbf{e}_y \to \mathbf{e}_y \cos t - \mathbf{e}_x \sin t$) Consider vector field

$$\mathbf{Y}(t) = \mathbf{e}_z + \mathbf{e}_y \cos t - \mathbf{e}_x \sin t$$

[15] it is just a special case of parallel transport considered in the section 4 above

.

Check that conditions

(229), (230) of proposition are satisfied.

1. $\mathbf{Y}(t)|_{t=0} = \mathbf{e}_z + \mathbf{e}_y = \mathbf{Y}$
Normal vector $\mathbf{n}(t)$ is equal to $\mathbf{r} = \mathbf{e}_x \cos t + \mathbf{e}_x \sin t$

$$(\mathbf{Y}(t), \mathbf{n}(t)) = (\mathbf{e}_z + \mathbf{e}_y \cos t - \mathbf{e}_x \sin t, \mathbf{e}_x \cos t + \mathbf{e}_y \sin t) = 0,$$

Hence condition (229) is satisfied.

$$\frac{d\mathbf{Y}(t)}{dt} = \frac{d}{dt}(\mathbf{e}_z + \mathbf{e}_y \cos t - \mathbf{e}_x \sin t) = -\mathbf{e}_x \cos t - \mathbf{e}_y \sin t = -\mathbf{n}(t)$$

Condition (230) is satisfied too. Hence it is indeed parallel transport.

Remark One can see that if we will do the parallel transport along a closed curve $\mathbf{r}(t), \mathbf{r}(a) = \mathbf{r}(b)$ then initial vector and the final vector will be different. E.g. consider close curved triangle ABC formed by three arcs of great circles: AB is the arc $x = \sin t, y = 0, z = \cos t, 0 \leq t \leq \frac{\pi}{2}$, BC is the arc $x = \cos t, y = \sin t, z = 0, 0 \leq t \leq \frac{\pi}{2}$ and CA is the arc $x = 0, y = \cos t, z = \sin t, 0 \leq t \leq \frac{\pi}{2}$

If the initial vector at the point A is $\mathbf{e}_y$, then its parallel transport at the point B will be again $\mathbf{e}_y$, parallel transport at the point C will be the vector $-\mathbf{e}_x$ and finally the returned vector at the point A will be the vector $- - \mathbf{e}_x$. The vector $\mathbf{e}_y$ transforms to the vector $-\mathbf{e}_x$. It is indication of the fact that there is no flat metric on the sphere [16]

6 Gaussian map and Gauss Bonnet Theorem

We try to give some ideas which lead to understanding and proof of the Gauss Bonnet Theorem (see the previous section).

6.1 Gaussian map

Let $M\colon\ \mathbf{r} = \mathbf{r}(u, v)$ be an oriented surface in $\mathbf{E}^3$ and $\mathbf{n}(u, v)$ normal vector filed. (Recall that at the every point $\mathbf{r}(u, v)$, the normal vector $\mathbf{n}(u, v)$ is orthogonal to the surface: $(\mathbf{n}, \mathbf{r}_u) = (\mathbf{n}, \mathbf{r}_v) = 0$) and the length of the vector $\mathbf{n}$ is equal to 1). Unit vector $\mathbf{n} = (n_x, n_y, n_z)$ can be considered as a point on the unit sphere $x^2 + y^2 + z^2 = 1$. Thus

[16] One can prove the following very beatiful formula: let γ be a closed curve which is a boundary of the domain on the surface M. Let $\mathbf{Y}(t_1)$ is parallel transport of the initial vector $\mathbf{Y}$ after travelling along the closed curve. Then the angle $\delta\varphi$ between these angles is equal to the integral of the Gaussian curvature of the surface over domain D:

$$\delta\varphi = \int_D K \sqrt{\det g}\, du dv \tag{231}$$

In particular for the sphere of radius R $\delta\varphi = \frac{\text{Area of } D}{R^2}$

we define a map which assigns to every point $p = \mathbf{r}(u, v) \in M$ the point on the unit sphere=unit vector $\mathbf{n}(u, v)$: This map is called Gaussian map.

Definition Gaussian map maps every point p of the oriented surface M to the normal unit vector $\mathbf{n}(p)$—point of the unit sphere:

$$\text{Gaussian map } \mathbf{n}\colon M \to S^2 \qquad M \ni \mathbf{r}(u, v) \mapsto \mathbf{n}(u, v) \in S^2 \tag{232}$$

The direction of normal vector is defined by the orientation of the surface.

Examples

1.Plane $Ax + By + Cz = 1$.

For all points of the plane unit normal vector $\mathbf{n}$ is the same vector. It is easy to see that

$$\mathbf{n} = \frac{1}{\sqrt{A^2 + B^2 + C^2}} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

The image of Gaussian map is just the point $\left(\frac{A}{\sqrt{A^2+B^2+C^2}}, \frac{B}{\sqrt{A^2+B^2+C^2}}, \frac{C}{\sqrt{A^2+B^2+C^2}}\right)$ on the unit sphere.

2. Cylinder $x^2+y^2 = a^2$. Normal vector $\mathbf{n}(h, \varphi)$ at the point $\mathbf{r}(h, \varphi) = (a \cos\varphi, a \sin\varphi, h)$ is equal to

$$\mathbf{n}(h, \varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

Gaussian map maps the cylinder on the equator points $(\cos\varphi, \sin\varphi, 0)$ of the sphere.

3. Upper part of the Cone: $x^2 + y^2 - k^2z^2 = 0$, $z \geq 0$.

The unit normal vector (see (160)) at the point $(kh\cos\varphi, kh\sin\varphi, h)$ is equal to

$$\mathbf{n} = \frac{1}{\sqrt{1 + k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

Gaussian map maps the upper part of the cone on the circle
$(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ on unit sphere where $\cos\theta = -\frac{k}{\sqrt{1+k^2}}, \sin\theta = \frac{1}{\sqrt{1+k^2}}$. (Another part of the cone $(z < 0)$ maps under Gaussian map to the circle $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, -\cos\theta)$)

4.Sphere $x^2+y^2+z^2 = R^2$. If $\mathbf{r}$ is the point on the sphere then unit vector is just equal to $\mathbf{r}/R$. Every point $\mathbf{r}$ of the sphere maps to the point $\mathbf{r}/R$ of the unit sphere. Gaussian map is one-one map.

Sphere is convex surface. It is a boundary of the ball which is convex body.

Consider arbitrary convex surface. (We call the closed surface convex if it is a boundary of convex domain. Domain D in $\mathbf{E}^3$ is called convex if for arbitrary two points $a, b \in D$ all the points of the interval $[a, b]$ belong to D).

One can see that Gaussian map establishes one-one correspondence between points of the surface M and points of unit sphere[17]

5. Torus: **(c)** Consider the torus M in $\mathbf{E}^3$ given by parameterisation:

$$\mathbf{r}(\varphi_1, \varphi_2) \colon \quad \begin{cases} x = (a + b\cos\varphi_1)\cos\varphi_2 \\ y = (a + b\cos\varphi_1)\sin\varphi_2 \\ z = b\sin\varphi_1 \end{cases},$$

where $0 \leq \varphi_1 < 2\pi$, $0 \leq \varphi_2 < 2\pi$ and a, b are constants such that $0 < b < a$.

One can prove that normal unit vector at the point $\mathbf{r}(\varphi_1, \varphi_2)$ is equal to

$$\mathbf{n}(\varphi_1, \varphi_2) = \begin{pmatrix} \cos\varphi_1 \cos\varphi_2 \\ \cos\varphi_1 \sin\varphi_2 \\ \sin\varphi_1 \end{pmatrix},$$

To prove it calculate the length of the vector $\mathbf{n}$ and prove that it is orthogonal to tangent vectors $\mathbf{r}_{\varphi_1}, \mathbf{r}_{\varphi_2}$. (Do it!)

Torus is not convex surface. Image of Gaussian map is whole unit sphere, but the map is not one-one correspondence. Every unit vector $\mathbf{n}$ (point of the unit sphere) has two pre images. E.g. consider $\mathbf{n} = (0, 1, 0)$. Then it follows from the previous formula that $\cos\varphi_1 \cos\varphi_2 = 0, \cos\varphi_1 \sin\varphi_2 = 1, \sin\varphi_1 = 0$. It implies two cases:

$\cos\varphi_1 = 1, \sin\varphi_1 = 0$, $\cos\varphi_2 = 0, \sin\varphi_2 = 1$, i.e. point on he torus $(0, a + b, 0)$

or $\cos\varphi_1 = -1, \sin\varphi_1 = 0$, $\cos\varphi_2 = 0, \sin\varphi_2 = -1$, i.e. point on he torus $(0, -(a - b), 0)$

In the first three cases (plane,cylinder, cone) Gaussian curvature of surfaces is equal to zero and image of Gaussian map is point (for plane) or curve (for cylinder and conus)

In the case of convex surface and torus image of Gaussian map is whole sphere.

In the case of convex surface the Gaussian map is one-one-correspondence. The Gaussian curvature at all the points is positive (see the footnote before (214)) and according to the Theorem the integral of curvature is equal to 4π. In the case of torus Gaussian map is not one-one-correspondence. The Gaussian curvature is positive , negative or equal to zero depending on the points of torus. Gauss-Bonnet Theorem tells that not only for torus but for every surface diffeomorphic to torus integral of gaussian curvature over the surface is equal to zero.

6.2 Gauß-Bonnet Theorem for convex surface

Here we give basic ideas to prove Gauß-Bonnet Theorem for convex surfaces. As it was noted in the example 4 the Gaussian map for these surfaces establishes one-one correspon-

[17] To prove this construct the map inverse to gaussian map on the unit sphere. Let $\mathbf{n}$ be an arbitrary unit vector. Consider a plane $l(\mathbf{n})$ tangent to this vector. It can be proved that there exists unique plane which is parallel to the plane $l(\mathbf{n})$ and which touches the surface M at same point p. It is easy to see that $\mathbf{n} = \mathbf{n}(p)$.

dence between points of surface and points of unit sphere.We prove that in this case:

$$\frac{1}{2\pi}\int_M K\sqrt{\det g}\,dudv = 2 \tag{233}$$

It is just Gauß-Bonnet Theorem for convex surfaces, because evidently convex surfaces are diffeomorphic to sphere (The convex domains are diffeomorphic to balls)

The proof of (233) in the case if Gaussian map establishes one-one correspondence between points of surface and points of unit sphere follows from

Proposition

Let M be an oriented surface in $\mathbf{E}^3$. Consider at arbitrary point p of this surface tangent vectors $\mathbf{a},\mathbf{b}$. Under the action of differential of Gaussian map vectors $\mathbf{a},\mathbf{b}$ transform to the vectors $\mathbf{a}',\mathbf{b}'$. Let $K(p)$ be gaussian curvature of the surface M at the point p. Then

$$K(p)S = S' \tag{234}$$

where S is an area of the parallelogram formed by the vectors $\mathbf{a},\mathbf{b}$, S' is an area of parallelogram formed by the vectors $\mathbf{a}',\mathbf{b}'$

Remark We consider the signed area of parallelogram. The modulus of area of parallelogram is the length of the cross product of vectors $\mathbf{a},\mathbf{b}$: $|S| = |\mathbf{a}\times\mathbf{b}|$. The signed area is defined by the direction of normal vector. It is equal to

$$S = (\mathbf{n}, \mathbf{a}\times\mathbf{b}) \tag{235}$$

If $\mathbf{n}\to-\mathbf{n}$, $S\to -S$. The direction of normal vector is defined by the orientation.

Show that this Proposition leads to the proof of (233).

Consider the covering of closed surface M by the collection $\{\Pi_k\}$ of infinitesimal parallelograms. Every infinitesimal parallelogram Π_k is attached to the point p_k of the surface and is formed by the tangent vectors $\mathbf{a}_k,\mathbf{b}_k$, i.e. it has sides $\varepsilon\mathbf{a}_k,\varepsilon\mathbf{b}_k$.

Consider Gaussian map of the surface M into S^2. Gaussian map establishes one-one correspondence. Hence under this map the covering of surface M by the collection $\{\Pi_k\}$ of infinitesimal parallelograms transforms onto the covering of unit sphere S^2 by the collection $\{\Pi'_k\}$ of infinitesimal parallelograms. According to (234)

$$\sum \text{Area of } \Pi'_k \approx \sum K(p_k)\text{Area of } \Pi'_k$$

These sums tend to corresponding integrals. Left hand side of this relation tends to the area of unit sphere. The right hand side of this relation tends to $\int_M K(p)\sqrt{\det g}\,dudv$. We come in the limit to the relation

$$4\pi = \text{area of unit sphere} = \int_M K(p)\sqrt{\det g}\,dudv$$

It is just (233)

Now we give a

Proof of the Proposition (234)

The proof of the Proposition follows from the following Lemma:

Lemma *In the vicinity of the given point p of the surface M consider unit normal vector field* $\mathbf{n}(u,v)$*, i.e. Gaussian map* (232)*. Then the action of differential* $d\mathbf{n}$ *on arbitrary tangent vector* $\mathbf{a}$ *is equal up to the sign to the action of the shape operator at this vector*

$$d\mathbf{n}(\mathbf{a}) = -S\mathbf{a} \tag{236}$$

(Shape operator at the point p (more precisely acting on the tangent vectors attached at the point p) is equal to $S = G^{-1}A$, where G is first quadratic form at the point p and A is second quadratic form at the point p, (see in detail Section 2))

The proof of this Lemma follows from definition of shape operator[18]

Show that Proposition follows from the Lemma.

Let vectors $\mathbf{a}, \mathbf{b}$ are attached to the point p. Let under the action of differential of Gaussian map the vector $\mathbf{a}$ transforms to the vector $\mathbf{a}'$ and vector $\mathbf{b}$ transforms to the vector $\mathbf{b}'$:

$$d\mathbf{n}(\mathbf{a}) = \mathbf{a}', \qquad d\mathbf{n}(\mathbf{b}) = \mathbf{b}$$

According to the Lemma:

$$\mathbf{a}' = -S(\mathbf{a}), \qquad \mathbf{b}' = -S(\mathbf{b}) \tag{237}$$

where $S = G^{-1}A$ is shape operator acting on the tangent vectors at the point p. Write down in components these relations. We will write vectors $\mathbf{a}, \mathbf{b}$, $\mathbf{a}', \mathbf{b}$ as columns 2×2 matrix:

$$(\mathbf{a}, \mathbf{b}) \to \begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix}, (\mathbf{a}', \mathbf{b}') \to \begin{pmatrix} a'^1 & b'^1 \\ a'^2 & b'^2 \end{pmatrix}, \text{shape operator } S = \begin{pmatrix} s^1_1 & s^1_2 \\ s^2_1 & s^2_2 \end{pmatrix},$$

Then relations (237) will have an appearance:

$$\begin{pmatrix} s^1_1 & s^1_2 \\ s^2_1 & s^2_2 \end{pmatrix} \begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix} = \begin{pmatrix} a'^1 & b'^1 \\ a'^2 & b'^2 \end{pmatrix} \tag{238}$$

Take determinant of this relation and use the fact that determinant is multiplicative: $\det(AB) = \det \cdot A \det B$. We come to the relation

$$\underbrace{\det \begin{pmatrix} s^1_1 & s^1_2 \\ s^2_1 & s^2_2 \end{pmatrix}}_{\text{I}} \cdot \underbrace{\det \begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix}}_{\text{II}} = \underbrace{\det \begin{pmatrix} a'^1 & b'^1 \\ a'^2 & b'^2 \end{pmatrix}}_{III} \tag{239}$$

Remembering the definition of Gaussian curvature we see that the first term is equal just to the Gaussian curvature at the point p: $K = \det S = \det(G^{-1}A) = \det A / \det G$.

Consider parallelogram Π formed by the vectors $\mathbf{a}, \mathbf{b}$ and parallelogram Π' formed by the vectors $\mathbf{a}', \mathbf{b}'$. Second and third determinants (up to a sign) are just areas of

[18]Let $\mathbf{a}' = d\mathbf{n}(\mathbf{a}) = a^\alpha \partial_\alpha n^i$. This vector is a vector tangent to M, because it is orthogonal to $\mathbf{n}$. Hence $\mathbf{a}' = a'^\alpha \mathbf{r}_\alpha = a^\alpha \partial_\alpha n^i$. Multiplying both parts by $\mathbf{r}_\beta$ we come to $a'^\alpha(\mathbf{r}_\alpha, \mathbf{r}_\beta) = a'^\alpha g_{\alpha\beta} = a^\alpha(\partial_\alpha n^i, \mathbf{r}_\beta)$. But $(\partial_\alpha n^i, \mathbf{r}_\beta) = -(n^i, \mathbf{r}_{\alpha\beta})$ because $(\mathbf{n}^i, \mathbf{r}_\beta) = 0$. Hence $a'^\alpha g_{\alpha\beta} = -a^\alpha A_{\alpha\beta}$. This leads to (236).

parallelograms Π, Π'. If we change the direction of normal vectors in (235) they both change a sign[19]. Hence the last relations is just:

$$K(p) \cdot \text{Area of the parallelogram } \Pi = \text{Area of the parallelogram } \Pi \tag{240}$$

It is just the statement of Proposition.

7 Appendices

7.1 Geodesics on the sphere and on Lobachevsky plane

In two-dimensional case the following lemma helps to find geodeics:

Lemma Consider metric which has the following appearance in the local coordinates u, v: $a(u)du^2 + b(u,v)dv^2$ where $a, b > 0$. Then for every curve $u(t), v(t)$, $t_0 \leq t \leq t_1$ the following inequality holds

$$\int_{t_0}^{t_1} \sqrt{a(u)u_t^2 + b(u,v)v_t^2}dt \geq \int_{t_0}^{t_1} \sqrt{a(u)u_t^2}dt = \tag{241}$$

$$\int_{t_0}^{t_1} \sqrt{a(u)}u_t dt = \int_{u_0}^{u_1} \sqrt{a(u)}du$$

where $u_0 = u(t_0), u_1 = u(t_1)$.

The proof of the lemma is obvious.

From this lemma it follows immediately that the lines $v = consta$ are geodesics of the metric $a(u)du^2 + b(u,v)dv^2$.

We can use this lemma to find geodesics of sphere and Lobachevsky plane.

Geodesics of sphere (See the subsection 3.3)

Consider riemannian metrics on the sphere in $\mathbf{E}^3$ with the radius a: Coordinates θ, φ, metrics (first quadratic form):

$$G = a^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{242}$$

Consider two arbitrary points A and B on the sphere. Let (θ_0, φ_0) be coordinates of the point A and (θ_1, φ_1) be coordinates of the point B

Let γ be a curve which connects these points: γ: $\theta(t), \varphi(t)$ such that $\theta(t_0) = \theta_0, \theta(t_1) = \theta_1$, $\varphi(t_0) = \theta_0, \theta(t_1) = \theta_1$ then:

$$L_{\gamma_{AB}} = \int a\sqrt{\theta_t^2 + \sin^2\theta(t)\varphi_t^2}dt \tag{243}$$

[19] to see that $(\mathbf{n}, [\mathbf{a}, \mathbf{b}]) = \pm\det\begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix}$ we note that left hand side and right hand side of this expression both are bilinear antisymmetric forms which coincide (up to a sign) on the vectors $\mathbf{a} = (1,0)$, $\mathbf{b} = (0,1)$

Without loss of generalisity suppose that they have the same latitude, i.e. if (θ_0, φ_0) are coordinates of the point A and (θ_1, φ_1) are coordinates of the point B then $\varphi_0 = \varphi_1$ (if it is not the fact then we can come to this condition rotating the sphere)

Now it is easy to see from the lemma that $\varphi = \varphi_0$ is geodesics: Indeed consider an arbitrary curve $\theta(t), \varphi(t)$ which connects the points A, B: $\theta(t_0) = \theta_0, \theta(t_1) = t_1$, $\varphi(t_0) = \varphi(t_1 = \varphi_0$. Compare its length with the length of the meridian which connects the points A, B:

$$\int_{t_0}^{t_1} a\sqrt{\theta_t^2 + \sin^2\theta\varphi_t^2}dt \geq a\int_{t_0}^{t_1} \sqrt{\theta_t^2}dt = a\int_{t_0}^{t_1} \theta_t dt = a(\theta_1 - \theta_0) \tag{244}$$

the big circles on sphere are geodesics. It corresponds to geometrical intuition: The geodesics on the sphere are the circles of intersection of the sphere with the plane which crosses the centre.

Lobachevsky plane and its geodesics

One of the model of Lobachevsky geometry is following: Consider upper plane of $\mathbf{E}^2$: (x, y) with $y \geq 0$ with metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{245}$$

The length of the curve γ: $x = x(t, y = y(t))$ is equal to

$$L = \int \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}}dt$$

In particularly the length of the vertical interval $[1, \varepsilon]$ tends to infinity if $\varepsilon \to 0$:

$$L = \int \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}}dt = \int_\varepsilon^1 \sqrt{\frac{1}{t^2}}dt = \log\frac{1}{\varepsilon}$$

One can see that the distance from every point to the line $y = 0$ is equal to infinity. This motivates the fact that the line $y = 0$ is called *absolute*.

It is easy to see from lemma that vertical lines are geodesics of Lobachevsky plane.

Find geodesics which connects two points A, B. Consider semicircle which passes these two points such that its centre is on the absolute.

We prove that it is a geodesic.

Proof Let coordinates of the centre of the circle are $(a, 0)$. Then consider polar coordinates (r, φ):

$$x = a + r\cos\varphi, y = r\sin\varphi \tag{246}$$

In these polar coordinates r-coordinate of the semicircle is constant.

Find Lobachevsky metric in these coordinates: $dx = -r\sin\varphi d\varphi + \cos\varphi dr$, $dy = r\cos\varphi d\varphi + \sin\varphi dr$, $dx^2 + dy^2 = dr^2 + r^2 d\varphi^2$. Hence:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dr^2 + r^2 d\varphi^2}{r^2\sin^2\varphi} == \frac{d\varphi^2}{\sin^2\varphi} + \frac{dr^2}{r^2\sin^2\varphi} \tag{247}$$

We see that the length of the arbitrary curve which connects points A, B is greater or equal to the length of the arc of the circle:

$$L_{AB} = \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2\varphi} + \frac{r_t^2}{r^2\sin^2\varphi}} dt \geq \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2\varphi}} dt = \tag{248}$$

$$\int_{t_0}^{t_1} \frac{\varphi_t}{\sin\varphi} dt = \int_{\varphi_0}^{\varphi_1} \frac{d\varphi}{\sin\varphi} = \log\frac{\tan\varphi_1}{\tan\varphi_1}$$

The proof is finished.

Why so much attention to Lobachevsky plane?

Lobachevsky plane appears as a first model of non-Euclidean geometry [20]

Why so much attention to Lobachevsky plane?

We already know Euclidean plane and sphere. They are two dimensional Riemannian manifolds with 3 isometries.(More exactly three-parametric group of isometries). Lobachevsky plane has three isometries too. And it is all!

In the class of two-dimensional Riemannian manifolds there are only these three cases with maximal group of symmetries. (It is easy to show that number of isometries cannot be more than $nn + 1/2$ for n-dimensional case.) So in some sense there are only three possibilities for geometry of two-dimensional manifold: usual euclidean, spheric and hyperbolic (geometry of Lobachevsky plane)

7.2 Surfaces of constant Gaussian curvatures in $\mathbf{E}^3$

We want to consider examples of surfaces of constant gaussian curvatures in $\mathbf{E}^3$. If $K > 0$ the best known example is sphere. Of course we can make a hole in sphere and .... it. (One can show that globally sphere cannot be ....)

If $K = 0$ there is again plenty trivial examples: plane, cylinder, cone,...

How to construct a surfaces with $K \equiv -1$. In other words how to realize Lobachevsky plane in $\mathbf{E}^3$.

We find the solution to this problem in the class of surfaces:

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = f(h)\cos\varphi \\ y = f(h)\sin\varphi \\ z = h \end{cases} \tag{249}$$

Calculate derivatives of $\mathbf{r}$ and normal vector $\mathbf{n}$:

$$\mathbf{r}_h = \begin{pmatrix} f_h\cos\varphi \\ f_h\sin\varphi \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -f\sin\varphi \\ f\cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{1+f_h^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -f_h \end{pmatrix}, \tag{250}$$

[20]Lobachevsky plane has the distinguished role from the point of view of fifth Euclid axiom.

$$\mathbf{r}_{hh} = \begin{pmatrix} f_{hh}\cos\varphi \\ f_{hh}\sin\varphi \\ 0 \end{pmatrix}, \quad \mathbf{r}_{h\varphi} = \begin{pmatrix} -f_h\sin\varphi \\ f_h\cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -f\cos\varphi \\ -f\sin\varphi \\ 0 \end{pmatrix}, \tag{251}$$

First quadratic form: metric is equal to:

$$G = \begin{pmatrix} (\mathbf{r}_h, \mathbf{r}_h) & (\mathbf{r}_h, \mathbf{r}_\varphi) \\ (\mathbf{r}_h, \mathbf{r}_\varphi) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} 1+f_h^2 & 0 \\ 0 & f^2 \end{pmatrix}, \quad G = (1+f_h^2)dh^2 + f^2(h)d\varphi^2 \tag{252}$$

and second quadratic form is equal to

$$A = \begin{pmatrix} (\mathbf{r}_{hh}, \mathbf{n}) & (\mathbf{r}_{h\varphi}, \mathbf{n}) \\ (\mathbf{r}_{h\varphi}, \mathbf{n}) & (\mathbf{r}_{\varphi\varphi}, \mathbf{n}) \end{pmatrix} = \frac{1}{\sqrt{1+f_h^2}} \begin{pmatrix} f_{hh} & 0 \\ 0 & -f \end{pmatrix} \tag{253}$$

Gaussian and Mean curvature are equal to:

$$K = \det(G^{-1}A) = \frac{\det A}{\det G} = \frac{-f_{hh}}{f(1+f_h^2)^2} \tag{254}$$

$$H = \mathrm{Tr}(G^{-1}A) = \frac{1}{\sqrt{1+f_h^2}} \left(\frac{f_{hh}}{1+f_h^2} - \frac{1}{f} \right) \tag{255}$$

To find revolution surfaces with constant (gaussian) curvatures we have to solve differential equation:

$$\frac{-f_{hh}}{f(1+f_h^2)^2} = K \tag{256}$$

It is evident from geometrical considerations that solution to this equation at $K > 0$ is sphere ($f = \sqrt{1-h^2}$) and its isometries.

In the case $K < 0$

7.3 On one beautiful formula

Let C be a surface in $\mathbf{E}^3$ $C\colon\quad \mathbf{r} = \mathbf{r}(u,v)$. let a surface C_w be *on the distance w from this surface*, i.e.

$$\mathbf{r}_w(u,v) = \mathbf{r}(u,v) + w\mathbf{n}(u,v), \tag{257}$$

where $\mathbf{n}(u,v)$ is a unit vectro onrthogonal to the surface C at the point $\mathbf{r}(u,v)$. (One can see that this vector will be orthogonal to the surface C_w too). There is a beautiful formula related Gaussian curvature of C_w with Gaussian and mean curvatures of C: If K, H be Gaussian curvatures of surface C at the point $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$ then Gaussian curvatture of the surface C_w at the point $\mathbf{r}_w(u_0, v_0) = \mathbf{r}(u_0, v_0) + w\mathbf{n}(u_0, v_0)$ is equal to

$$K = \frac{K}{1 - wH + w^2K} \tag{258}$$

In particularly

if the surface C has the constant mean curvature $H \equiv h$ then the surface C_w which is in the distance $w = \frac{1}{h}$ from the surface C has a constant Gaussian curvature equal to h.

The proof is founded on the formulae for gaussian curvature and very elementary use of Cayley-Hamilton identities.

Namely if $x^i = x^i(u_\alpha) + wn^i(u_a lpha)$ (it is conveninet to denote by x^i components of vector $\mathbf{r}$, $(i = 1, 2, 3)$, by u_α $(\alpha = 1, 2)$ parameteres u, v). Denote by $g(w)_{\alpha\beta} A(w)\alpha, \beta$ tensors of metric (first quadratic form) and second quadratic form. Note that vector $\mathbf{n}$ is orthogonal to the surface C as well: $(\mathbf{r}_\alpha + w\mathbf{n}_\alpha, \mathbf{n}) = 0$ because

$$(\mathbf{n}_a, \mathbf{n}) = n^i_\alpha n^i = \frac{1}{2}\frac{\partial}{\partial u^\alpha}\left(n^i n^i\right) = 0\,. \tag{259}$$

Then:

$$g_{\alpha\beta}(w) = (\mathbf{r}_\alpha + w\mathbf{n}, \mathbf{r}_\beta + w\mathbf{n}) = (x^i_\alpha + wn^i_\alpha)(x^i_\beta + wn^i_\beta), \quad A_{\alpha\beta}(w) = (\mathbf{r}_{\alpha\beta} + w\mathbf{n}_{\alpha\beta}, \mathbf{n}) \tag{260}$$

It is easy to see from definition of $\mathbf{n}$ and (259) that vector $\mathbf{n}_a$ is tangent to the surface C and the following relations hold:

$$(\mathbf{r}_\alpha, \mathbf{n}_\beta) = x^i_\alpha n^i \beta = -x^i_{\alpha\beta} n^i = -A_{\alpha\beta}, \qquad (\mathbf{n}_a, \mathbf{n}_\beta) = \left(A \cdot g^{-1} \cdot A\right)_{\alpha\beta} \tag{261}$$

Now everything is ready to calculate $K(w)$:

$$K(w) = \frac{\det A(w)}{\det g(w)} = \frac{\det\left(A_{\alpha\beta} + wn^i_{\alpha\beta} n^i\right)}{\det\left(g_{\alpha\beta} + 2wx^i_\alpha n^i_\beta + w^2 n^i_\alpha n^i_\beta\right)} =$$

$$\frac{\det\left(A_{\alpha\beta} - wn^i_\alpha n^i_\beta\right)}{\det\left(g_{\alpha\beta} - 2wx^i_{\alpha\beta} n^i + w^2 n^i_\alpha n^i_\beta\right)} = \frac{\det\left(A - wAg^{-1}A\right)}{\det\left(g - 2wA + w^2 Ag^{-1}A\right)}$$

Remember that

$$K = \det S = \frac{\det A}{\det G}, H = \operatorname{Tr} S, \quad where\ S = g^{-1}A$$

Also it is useful to use the following identity for 2×2 matrices:

$$\det(1 + A) = 1 + \operatorname{Tr}\, A + \det A \tag{262}$$

It is the elementariest of Cayley-Hamilton identities.

Now using this we can calculate $K(w)$:

$$K(w) = \frac{\det\left(A - wAg^{-1}A\right)}{\det\left(g - 2wA + w^2 Ag^{-1}A\right)} = \frac{\det A \det\left(1 - wg^{-1}A\right)}{\det g \det\left(1 - 2wg^{-1}A + w^2 g^{-1}Ag^{-1}A\right)} \tag{263}$$

$$= \frac{\det A \det\left(1 - wg^{-1}A\right)}{\det g \det^2\left(1 - wg^{-1}A\right)} = \frac{\det A}{\det g \det\left(1 - wS\right)} = \frac{K}{1 - wH + w^2 K} \tag{264}$$

7.4 Tubes

The ideas of the previous Appendix can be developed.

Consider the function $\det(1 + zS)$, where z is formal parameter. This function is quadratic polynomial in z and coefficients are just mean and Gaussian curvature:

$$\det(1 + zS) = 1 + Hz + Kz^2 \tag{265}$$

because for operators in 2-dimensional space $\det(1 + A) = 1 + \mathrm{Tr}A + detA$

If M is a compact surface in $\mathbf{E}^3$ then one can consider remarkable polynomial:

$$P_M(z) = \int_M \det(1 + zS)\sqrt{g}d^2x \tag{266}$$

Here $\sqrt{g}d^2x$ is volume form defined by the first quadratic form. On one hand monoms of this polynomial defines invariants of the surface M, on the other hand for small z this polynomial defines the area of the surface M_z which is on the distance z from M:

$$S(M_z) = \underbrace{\int_M 1 \cdot \sqrt{g}d^2x}_{S(M_0)} + z \underbrace{\int_M H\sqrt{g}d^2x}_{S(M)\cdot\text{averaged mean curvature}} + \underbrace{\int_M K\sqrt{g}d^2x}_{2\pi\cdot(\text{Euler number of } M)} \tag{267}$$

One cna prove thisformula in the following way: Consider in tubular neighbourhood of M coordinates (ξ^1, ξ^2, ρ) such that ρ measures he distance from the points till M One can see that This formula has many applications.

In particular it allows to calculate an average mean and gaussian curvature for singular surfaces: parallelepiped. It is evident that if M is parallelepiped with sides $a, b, c,$. Then

$$S(M_z) = 2(ab + ac + bc) + \pi z(a + b + c) + 4\pi z^2 \tag{268}$$

Comparing these formulae we see that averaged mean curvature of parallelepiped is equal to $\frac{\pi(a+b+c)}{2(ab+ac+bc)}$

These formulaes can be easy generalised for hypersurfaces in $\mathbf{E}^n$

7.5 Levi Civita connection II

Let M be a Riemannian manifold with metrics G.

Recall that metrics defines scalar product of vector fields:

$$< \mathbf{X}, \mathbf{Y} >= g_{ik}X^iY^k$$

Let ∇ be a connection on M:

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^i \frac{\partial Y^k}{\partial x^i} + X^iY^k\Gamma^m_{ik}$$

We say that this connection is *symmetric connection* if Christophel symbol Γ^m_{ik} satisfies the condition [21]:

$$\Gamma^m_{ik} = \Gamma^m_{ki}$$

We say that symmetric connection ∇ is Levi Civita connection if it preserves scalar, i.e. if for arbitrary vectors $\mathbf{Y}, \mathbf{Z}$ attached at the arbitrary point the

$$\nabla_{\mathbf{X}} < \mathbf{Y}, \mathbf{Z} > = < \nabla_{\mathbf{X}}(\mathbf{Y}), \mathbf{Z} > + < \mathbf{Y}, \nabla_{\mathbf{X}}(\mathbf{Z}) > \tag{269}$$

Theorem On the Riemannian manifold (M, G) there exists uniquely defined Levi Civita connection. In local coordinates Christoffel symbols of this connection have the following appearance:

$$\Gamma^m_{ik}(x) = \frac{1}{2} g^{mn}(x) \left(\frac{\partial g_{in}(x)}{\partial x^k} + \frac{\partial g_{kn}(x)}{\partial x^i} - \frac{\partial g_{ik}(x)}{\partial x^n} \right) \tag{270}$$

Prove this Theorem. Suppose that there exist symmetric connection ∇ satisfying condition with Christoffel symbols Γ^m_{ik} in local coordinates. We show that these coefficients are defined uniquely by the condition (269).

Rewrite the condition (269) in components for $\mathbf{X} = \mathbf{e}_m$:

$$\partial_m \left(g_{ik} Y^i Z^k \right) = g_{ik} \left(\partial_m Y^i + \Gamma^i_{mr} Y^r \right) Z^k + g_{ik} Y^i \left(\partial_m Z^k + \Gamma^k_{mr} Z^r \right) \tag{271}$$

Comparing left and right hand sides of this expression for arbitrary vectors $\mathbf{Y}, \mathbf{Z}$ we see that

$$\partial_m g_{ik} = \Gamma_{k;mi} + \Gamma_{i;mk} \tag{272}$$

where we denote by

$$\Gamma_{k;mi} = g_{kr} \Gamma^r_{mi}$$

Now using the symmetricity condition $\Gamma^r_{mi} = \Gamma^r_{im}$ we obtain that

$$\Gamma_{i;mk} = \partial_m g_{ik} - \Gamma_{k;im} = \partial_m g_{ik} - (\partial_i g_{km} - \Gamma_{m;ik}) = \partial_m g_{ik} - \partial_i g_{km} + \partial_k g_{im} - \Gamma_{i;mk}$$

Hence

$$\Gamma_{i;mk} = \frac{1}{2} \left(\partial_m g_{ik} + \partial_k g_{im} - \partial_i g_{km} \right), \quad \Gamma^i_{mk} = g^{ij} \Gamma_{j;mk} \tag{273}$$

and we come to (270).

One can see that Levi Civita connection is well defined by Christophel symbols (270).

Example Consider two-dimensional surface with Riemannian metrics

$$G = a(u,v) du^2 + b(u,v) dv^2, \qquad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a(u,v) & 0 \\ 0 & b(u,v) \end{pmatrix}$$

[21] In a more invariant way one can define define a symmetric connection ∇ as a connection which satisfies the condition:

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0$$

The left hand side of the formula above defines the torsion of the connection.

Calculate Christoffel symbols of Levi Civita connection.

Using (273) we see that:

$$\begin{aligned}
\Gamma_{1;11} &= \tfrac{1}{2}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) = \tfrac{1}{2}\partial_1 g_{11} = & \tfrac{1}{2}a_u \\
\Gamma_{1;21} = \Gamma_{1;12} &= \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}\right) = \tfrac{1}{2}\partial_2 g_{11} = & \tfrac{1}{2}a_v \\
\Gamma_{1;22} &= \tfrac{1}{2}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) = -\tfrac{1}{2}\partial_1 g_{22} = - & \tfrac{1}{2}b_u \\
\Gamma_{2;11} &= \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}\right) = -\tfrac{1}{2}\partial_2 g_{11} = - & \tfrac{1}{2}a_v \\
\Gamma_{2;12} = \Gamma_{2;21} &= \tfrac{1}{2}\left(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{21}\right) = \tfrac{1}{2}\partial_1 g_{22} = & \tfrac{1}{2}b_u \\
\Gamma_{2;22} &= \tfrac{1}{2}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) = \tfrac{1}{2}\partial_2 g_{22} = & \tfrac{1}{2}b_v
\end{aligned} \tag{274}$$

To calculate $\Gamma^i_{km} = g^{ir}\Gamma_{r;km}$ note that for the metric $a(u,v)du^2 + b(u,v)dv^2$

$$G^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{a(u,v)} & 0 \\ 0 & \frac{1}{b(u,v)} \end{pmatrix}$$

Hence

$$\begin{aligned}
\Gamma^1_{11} = \; & g^{11}\Gamma_{1;11} = \; \tfrac{a_u}{2a}, \quad & \Gamma^1_{21} = \Gamma^1_{12} = \; & g^{11}\Gamma_{1;12} = \; \tfrac{a_v}{2a}, \quad & \Gamma^1_{22} = \; & g^{11}\Gamma_{1;22} = \; \tfrac{-b_u}{2a} \\
\Gamma^2_{11} = \; & g^{22}\Gamma_{2;11} = \; \tfrac{-a_v}{2b}, \quad & \Gamma^2_{21} = \Gamma^2_{12} = \; & g^{22}\Gamma_{2;12} = \; \tfrac{b_u}{2b}, \quad & \Gamma^2_{22} = \; & g^{22}\Gamma_{2;22} = \; \tfrac{b_v}{2b}
\end{aligned} \tag{275}$$

Example Sphere.

On the sphere first quadratic form (Riemannian metric) $G = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2$ Hence we use calculations from previous example with $a(\theta,\varphi) = R^2, b(\theta,\varphi) = R^2\sin^2\theta$ $(u = \theta, v = \varphi)$. Note that $a_\theta = a_\varphi = b_\varphi = 0$. Hence only non-trivial components of Γ will be:

$$\Gamma^1_{22} = \frac{-b_\theta}{2a} = \frac{-\sin 2\theta}{2}, \qquad \left(\Gamma_{1;22} = \frac{-R^2\sin 2\theta}{2}\right), \tag{276}$$

$$\Gamma^2_{12} = \frac{b_\theta}{2b} = \frac{\cos\theta}{\sin\theta} \qquad \left(\Gamma_{2;12} = \frac{R^2\sin 2\theta}{2}\right) \tag{277}$$

All other components are equal to zero:

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} = 0$$

7.6 Riemannian curvature

Let M be manifold with connection ∇.

Then one can consider the *curvature* of this connection defined by the relation:

$$R(\mathbf{X},\mathbf{Y}) = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla\mathbf{X} - \nabla_{[\mathbf{X},\mathbf{Y}]} \tag{278}$$

$R(\mathbf{X}, \mathbf{Y})$ defines operation on vector field.

$$(R(\mathbf{X}, \mathbf{Y})\mathbf{Z})^i = Z^k R^i_{kpq} X^p Y^q \tag{279}$$

where

$$R^i_{kpq} = \partial_p \Gamma^i_{qk} - \partial_q \Gamma^i_{pk} + \Gamma^i_{ps}\Gamma^s_{qk} - \Gamma^i_{qs}\Gamma^s_{pk} \tag{280}$$

is (1.3) tensor.

Christoffel symbol is not a tensor. The object defined above is *a tensor*.

In particularly if $R^i_{kpq} \equiv 0$ in given local coordinates then it is equal to zero in every local coordinates. This gives constructive answer to the question:

Question: Consider the Riemannian manifold with metric $G = g_{ik}(u)du^i dv^k$. How to know do there exis new coordinates such that in these new coordinates metric is flat:

$$G = g_{ik}(u)du^i dv^k = g_{ik}\left(u(x)\right)\frac{\partial u^i(x)}{dx^a}\frac{\partial u^k(x)}{dx^b}dx^a dx^b = (dx^1)^2 + \cdots + (dx^n)^2$$

To answer this question one have to calculate Levi-Civita connection (270) of the metric G then the Riemann tensor (280) of this connection. Then:

Theorem Riemann tensor of Levi-Civita connection $\Gamma(g)$ is equal to zero (in a vicinity of the point) iff there are local coordinates (in a vicinity of this point) such that metric in these local coordinates is cartesian.

We will partly to discuss this for two-dimensional manifolds in the next subsections.

7.7 Scalar curvature. Gauss *Theorema Egregium*

Let M be Riemannian manifold with metric G. Let ∇ be Levi-Civita connection of this metric R^i_{kpq} be Riemann curvature tensor.

Consider tensor $R_{ikpq} = g_{ij}R^j_{kpq}$. One can show that tensor R_{ikpq} obeys the following properties:

$$R_{ikpq} = -R_{kipq}, R_{ikpq} = -R_{ikqp}, R_{ikpq} = R_{pqki} \tag{281}$$

One can consider scalar curvature:

$$R = g^{ip}g^{kq}R_{ikpq}$$

It is a scalar which is called scalar curvature.

In the case of 2-dimensional space formulae are extremely simple: Tensor R_{ikpq} has only one non-trivial components which we will denote by p:

$$R_{1212} = -R_{2112} = R_{2121} = P$$

Indeed $R_{1112} = R_{2212} = R_{1122} = R_{1211} = \cdots = 0$ by the condition (281). Scalar curvature in this case is equal to:

$$R = g^{ip}g^{kq}R_{ikpq} = g^{11}g^{22}R_{1212} + g^{12}g^{21}R_{1221} + g^{21}g^{12}R_{2112} + g^{22}g^{11}R_{2121} =$$

$$P\left(g^{11}g^{22} - g^{12}g^{21} - g^{21}g^{12} + g^{22}g^{11}\right) = 2P(g^{11}g^{22} - g^{12}g^{21}) =$$

$$\frac{2p}{\det G} = \frac{2R_{1212}}{g_{11}g_{22} - g_{12}^2}$$

On two dimensional surface embedded in $\mathbf{E}^3$ one can consider Riemannian metric defined by first quadratic form: $G = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2$, where $g_{11} = (\mathbf{r}_u, \mathbf{r}_u), g_{12} = (\mathbf{r}_u, \mathbf{r}_v), g_{22} = (\mathbf{r}_v, \mathbf{r}_v)$, Levi-Civita connection ∇ with Christopher symbols defined by (270) and curvature defined by (280). Emphasize again that in the case of two-dimensional surface there exist only one non-trivial component of riemannian tensor.

Now we formulate

Theorema Egregium *Let C be two-dimensional surface in $\mathbf{E}^3$. Let G be Riemannian metric on this surface defined by the first quadratic form. Let R be scalar curvature defined by the Riemann curvature tensor and K Gaussian curvature. Then*

$$\frac{R}{2} = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = K \tag{282}$$

In the right hand side of this formula stands Gaussian curvature. It is defined by the way of how surface is embedded in $\mathbf{E}^3$. External observer calculates second and first quadratic form and obtains Gaussian curvature (See the Section 2). In the left hand side of this formula stands Riemannian scalar curvature. It is defined by the metric on the surface. Internal observer, aunt on the surface, takes the metric [22] and calculates curvature without any knowledge of the second quadratic form. The Theorem states that the two answers will coincide

In particular if we bend the surfaces (i.e. transform it without changing the metric, then r.h.s. will be the same. Hence Gaussian curvature will be the same.

The Theorem above explains why it is not possible to consider on the sphere coordinates u, v such that $g = du^2 + dv^2$, i.e. it is not possible to bend the plane list to the sphere. Indeed suppose there exist $u, v\colon \theta = \theta(u, v), \varphi = \varphi(u, v)$ such that

$$R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2 = du^2 + dv^2 \tag{283}$$

Then scalar curvature R is equal to zero. But R.H.S. of (282) is Gaussian curvature of the sphere. It is equal to $\frac{1}{R^2}$. Contradiction.

This formula in particlularly enables to calculate Riemannian curvature for two-dimensional surfaces in $\mathbf{E}^3$. E.g. for sphere $K = \frac{1}{R^2}$, $\det g = R^4 \sin^2\theta$. Hence $R_{1212} = K \det g = R^2 \sin^2\theta$. On the other hand the same answer follows from the straightforward calculations of R_{1212} via formulae (269) and (280).

7.8 A Tale on Differential Geometry

Once upon a time there was an ant living on a sphere of radius R. One day he asked himself some questions: What is the structure of the Universe (surface) where he lives? Is

[22] which of course is defined by the external observer because metric is defined by the first quadratic form. But external observer send information to aunt only about metric not about second quadratic form!

it a sphere? Is it a torus? Or may be something more sophisticated, e.g. pretzel (a surface with two holes)

Three-dimensional human beings do not need to be mathematicians to distinguish between a sphere torus or pretzel. They just have to look on the surface. But the ant living on two-dimensional surface cannot fly. He cannot look on the surface from outside. How can he judge about what surface he lives on [23]?

Our ant loved mathematics and in particular *Differential Geometry.* He liked to draw various triangles, calculate their angles α, β, γ, area $S(\Delta)$. He knew from geometry books that the sum of the angles of a triangle equals π, but for triangles which he drew it was not right!!!!

Finally he understood that the following formula is true: For every triangle

$$\frac{(\alpha+\beta+\gamma-\pi)}{S(\Delta)} = c \tag{1}$$

A constant in the right hand side depended neither on size of triangle nor the triangles location. After hard research he came to conclusion that its Universe can be considered as a sphere embedded in three-dimensional Euclidean space and a constant c is related with radius of this sphere by the relation

$$c = \frac{1}{R^2} \tag{2}$$

...Centuries passed. Men have deformed the sphere of our old ant. They smashed it. It seized to be round, but the ant civilisation survived. Moreover old books survived. New ant mathematicians try to understand the structure of their Universe. They see that formula (1) of the Ancient Ant mathematician is not true. For triangles at different places the right hand side of the formula above is different. Why? If ants could fly and look on the surface from the cosmos they could see how much the sphere has been damaged by humans beings, how much it has been deformed, But the ants cannot fly. On the other hand they adore mathematics and in particular *Differential Geometry.* One day considering for every point very small triangles they introduce so called curvature for every point P as a limit of right hand side of the formula (1) for small triangles:

$$K(P) = \lim_{S(\Delta)\to 0} \frac{(\alpha+\beta+\gamma-\pi)}{S(\Delta)}$$

Ants realise that curvature which can be calculated in every point gives a way to decide where they live on sphere, torus, pretzel... They come to following formula [24] : integral of curvature over the whole Universe (the sphere) has to equal 4π, for torus it must equal 0, for pretzel it equalts -4π...

$$\frac{1}{2\pi}\int K(P)dP = 2\,(1 - \text{number of holes})$$

[23]This is not very far from reality: For us human beings it is impossible to have a global look on three-dimensional manifold. We need to develop local methods to understand global properties of our Universe. *Differential Geometry* allows to study global properties of manifold with local tools.

[24]In human civilisation this formula is called Gauß-Bonet formula. The right hand side of this formula is called Euler characteristics of the surface.