Jacobi identity and intersection of altitudes

It is many years that I know the expression which belongs to Arnold and which sound something like that: "Altitudes (heights) of triangle intersect in one point because of Jacoby identity". or may be even more aggressive: "The geometrical meaning of Jacoby is contained in the fact that altitudes of triangle are intersected in the one point". Today preparing exercises for students I suddenly understood a meaning of this sentence. Here it is:

Let ABC be a triangle. Denote by $\mathbf{a}$ vector BC, by $\mathbf{b}$ vector CA and by $\mathbf{c}$ vector AB: $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$. Consider vectors $\mathbf{N_a}=[\mathbf{a},[\mathbf{b},\mathbf{c}]]$, $\mathbf{N_b}=[\mathbf{b},[\mathbf{c},\mathbf{a}]]$ and $\mathbf{N_c}=[\mathbf{c},[\mathbf{a},\mathbf{b}]]$. Vector $\mathbf{N_a}$ applied at the point A of the triangle ABC belongs to the plane of triangle, it is perpendicular to the side BC of this triangle. Hence the altitude (height) h_A of the triangle which goes via the vertex A is the line h_A: $A+t\mathbf{N_a}$. The same is for vectors $\mathbf{N_b},\mathbf{N_c}$: Altitude (height) h_B is a line which goes via the vertex B along the vector $\mathbf{N}_b$ and altitude h_C (height) is a line which goes via the vertex C along the vector $\mathbf{N}_c$.

Due to Jacobi identity sum of vectors $\mathbf{N_a},\mathbf{N_b},\mathbf{N_c}$ is equal to zero:

$$\mathbf{N_a}+\mathbf{N_b}+\mathbf{N_c}=[\mathbf{a},[\mathbf{b},\mathbf{c}]]+[\mathbf{b},[\mathbf{c},\mathbf{a}]]+[\mathbf{a},[\mathbf{b},\mathbf{c}]]=0 \tag{1}$$

To see that altitudes h_A: $A+t\mathbf{N_a}$, h_B: $B+t\mathbf{N_b}$ and h_C: $C+t\mathbf{N_c}$ intersect in one point it is enough to show that the sum of torques (angular momenta) of vectors $\mathbf{N_a}$ at the line h_A, $\mathbf{N_b}$ at the line h_B and $\mathbf{N_c}$ at the line h_C vanishes with respect to at least one point M:

$$[MA,\mathbf{N_a}]+[MB,\mathbf{N_b}]+[MC,\mathbf{N_c}]=0, \tag{2}$$

because sum of these vectors is equal to zero. Indeed note that if relation (2) obeys for any given point Mthen it obeys for an arbitrary point M' because of relation (1). Suppose lines l_A, l_B intersect at the point O. Take a point O instead a point M in the relation (2). Then $[OA,\mathbf{N_a}]=[OB,\mathbf{N_b}]=0$. Hence $[OC,\mathbf{N_c}]=0$, i.e.point O belongs to the line l_C too. Hence it suffices to show that relation (2) is satisfied. We again will use Jacobi identity: Take an arbitrary point M. Denote $MA=\mathbf{x}$ then for left hand side of the equation (2) we have $[MA,\mathbf{N_a}]+[MB,\mathbf{N_b}]+[MC,\mathbf{N_c}]=[\mathbf{x},\mathbf{N_a}]+[\mathbf{x}+\mathbf{c},\mathbf{N_b}]+[\mathbf{x}+\mathbf{c}+\mathbf{a},\mathbf{N_c}]=[\mathbf{c},\mathbf{N_b}]+[\mathbf{c}+\mathbf{a},\mathbf{N_c}]$ (due to (1)). Now $[\mathbf{c},\mathbf{N_b}]+[\mathbf{c}+\mathbf{a},\mathbf{N_c}]=[\mathbf{c},\mathbf{N_b}]-[\mathbf{b},\mathbf{N_c}]$ and $[\mathbf{c},\mathbf{N_b}]-[\mathbf{b},\mathbf{N_c}]=[\mathbf{c},[\mathbf{b},[\mathbf{c},\mathbf{a}]]]-[\mathbf{b},[\mathbf{c},[\mathbf{a},\mathbf{b}]]]$. But $[\mathbf{a},\mathbf{b}]=[\mathbf{a},-\mathbf{a}-\mathbf{c}]=[\mathbf{c},\mathbf{a}]$ since $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$. Hence and here we again will use Jacoby identity:

$$[\mathbf{c},[\mathbf{b},[\mathbf{c},\mathbf{a}]]]-[\mathbf{b},[\mathbf{c},[\mathbf{a},\mathbf{b}]]]=[\mathbf{c},[\mathbf{b},[\mathbf{c},\mathbf{a}]]]-[\mathbf{b},[\mathbf{c},[\mathbf{c},\mathbf{a}]]]=[[\mathbf{c},\mathbf{a}],[\mathbf{c},\mathbf{b}]]=[[\mathbf{c},\mathbf{a}],[\mathbf{c}+\mathbf{a},\mathbf{c}]]=0\blacksquare$$

Hence altitudes of triangle intersect in one point! Zabavno, da? ■

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