Notes On Making Mathematical Notes For Your Course

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This is a summary of guidance notes for students on mathematical language, symbols, logic and proofs, and on how to make their own course notes useful. It suggests some essential skills that a student should acquire during a first semester at university.

Language, symbols, logic and proofs

1. **Language**

   (i) Make sense to a computer (*not* a telepath) using same axioms.

   (ii) Punctuate into sentences and paragraphs. Brackets are useful.

   (iii) Signpost arguments : before, during and after the development.

2. **Symbols**

   (i) Declare the sets in use; eg $A = \{x \in \mathbb{R}^m | (\text{statement about } x)\}$.

   (ii) Introduce an element by declaring its origin and make clear how it was chosen or if arbitrary, from which set.

   (iii) Use '=' as an abbreviation of aequare, 'to equal' (Descartes introduced it as α in the 17^{th} century). It can be used *only* between sets, or between elements of one set; *not* as an abbreviation of 'is'.

   (iv) Use $\leq$ only between real numbers (or elements of posets, later).

   (v) Use $\Longrightarrow$ only between statements; not to begin a sentence.

   (vi) Use brackets for clarity, eg to separate quantified phrases:
   $(\forall \epsilon > 0)(\exists \delta > 0) : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$

3. **Logic**

   (i) **If** . . . , **then** . . . . (Follow 'if' with 'then'.)

   (ii) Learn how to read and negate ($\neg$) statements:
   $(P \Rightarrow Q) \Longleftrightarrow (\neg Q \Longrightarrow \neg P)$
   Try these statements in words and symbols; note the way that 'and' and 'or' arise as mutual negations:
   Statement S_1 : All lecturers are hairy **or** have a glass eye.
   Symbolically S_1 : $(\forall x \in L)$ $H(x)$ **or** $G(x)$.
   Negation $\neg S_1$: $(\exists x \in L)$: $\neg H(x)$ **and** $\neg G(x)$.
   Note the difference between S_1 *and the following* S_2 :
   Statement S_2:
   All lecturers are hairy **or** all lecturers have a glass eye.
   Symbolically S_2 : $(\forall x \in L,\ H(x))$ **or** $(\forall x \in L,\ G(x))$.
   Negation $\neg S_2$: $(\exists x \in L : \neg H(x))$ **and** $(\exists y \in L : \neg G(y))$.

4. **Proofs**

   (i) A common error is to assume the result and show it is reasonable.

   (ii) Clearly declare definitions you need.

   (iii) Know a standard layout for each of these seven situations;

   (1) $P \Rightarrow Q$. : −
   Either, directly
   assume P is true and then deduce that Q must be true.
   Or use the equivalence $(P \Rightarrow Q) \Longleftrightarrow (\neg Q \Longrightarrow \neg P)$
   assume Q is false and then deduce that P must be false.

(2) $P \iff Q$. :− (a) $P \Rightarrow Q$ **and** (b) $Q \Rightarrow P$.
(3) $A = B$. :− (a) $A \subseteq B$ **and** (b) $B \subseteq A$.
(4) $(\forall x \in L)\, Q(x)$. :− Take **arbitrary** $x \in L, ...$ deduce $Q(x)$.
(5) $(\exists x \in L) : Q(x)$. :− Find **any** $x \in L$, with $Q(x)$ true.
(6) $(\exists! x \in L) : Q(x)$. :− Do (5). Suppose also x^1 satisfies (5). Show uniqueness of x by proving $x = x^1$.
(7) Induction :- Show P_1 true. Assume P_k true for an **arbitrary** $k \geq 1$. Show P_{k+1} true.

Making lecture notes useful

1. **Definitions and theorems**

   (i) Augment your notes with examples, and especially with crucial non-examples. You are expected to find these for yourself as well, e.g. in books and tutorials.
   (ii) Definitions often arise out of crucial non-examples, and the obscure parts of proofs are frequently needed to circumvent non-obvious pathological cases. Find out which examples broke the teeth of previous theories and why.
   (iii) Some definitions are 'natural', for example: a metric space and a dual vector space. Find out why.
   (iv) An index for your notes is worth having.

2. **Tutorial problems**

   (i) Read your (augmented) lecture notes around the definitions and check other sources (more than one book).
   (ii) Decide which layout under 4(iii) above that you need.
   (iii) If difficult to begin, then first try a simplified problem.
   (iv) Most problems appear as examples or theorems in some book.
   (v) Consult your tutor if a reasonable attempt fails.

3. **Revision**

   (i) Reading proofs is only for insomniacs.
   (ii) For revision, write out illustrative examples in parallel with the proofs. Contrast non-examples.
   (iii) Use a range of books not just one. Use your tutors.