Mathematics 351: Background Notes and Exercises

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An important part of studying pure mathematics is learning to recognise mathematical structures, the way they arise and how they are classified; in this course we encounter a number of such structures, from algebra, topology and algebraic topology. These notes supplement the lectures and provide practise exercises for revision and new topics. We begin with some material you will have met before, perhaps in other forms, to set some terminology and notation, giving definitions for items in boldface. You should at least read all of the exercises in the sections you need and try as many as you have time for. Some topics may be unfamiliar and are not central to this course, for example, category theory, but that is likely to be met in any further study of geometry, topology or algebra and is an important framework for mathematical structures.

Further details on unfamiliar topics may be found in Armstrong [2] for beginning topology, Cohn [3] for algebra, Dodson and Poston [5] for linear algebra, topology and differential geometry, Dodson and Parker [4] for algebraic topology, Gray [6] for curves, surfaces and calculations using the computer algebra package *Mathematica*, and Wolfram [10] for *Mathematica* itself. Several on-line hypertext documents are provided for this course via the web server [1].

1 Sets and maps

We recall some basic definitions.

1.1 Definitions

Let X and Y be non-empty sets. A **relation** from X to Y is a subset $\rho \subseteq X \times Y$, which means that it can be represented equivalently by its **graph** in the $X - Y$ space. We write $x\rho\, y$ if $(x, y) \in \rho$ and define for ρ its **domain** dom $\rho = \{x \in X \mid \exists\, y \in Y \text{ with } (x, y) \in \rho\}$ and its **image** im $\rho = \{y \in Y \mid \exists\, x \in \text{dom}\,\rho \text{ with } (x, y) \in \rho\}$. When dom $\rho = X$ we say that ρ is an **entire** relation; we shall use *only* entire relations so we shall not need this qualification. A relation $\rho \subseteq X \times X$ may have any or none of the following properties:

symmetry	$x\rho\, y$ if and only if $y\rho\, x$
reflexivity	for all $x \in X$, $x\rho\, x$
transitivity	for all $x, y, z \in X$, $x\rho\, y$ and $y\rho\, z$ implies $x\rho\, z$
equivalence	symmetry, reflexivity, and transitivity
antisymmetry	$x\rho\, y$ and $y\rho\, x$ implies $x = y$
partial order	antisymmetry, reflexivity, and transitivity.
total order	for all $x, y \in X$, either $x\rho\, y$ or $y\rho\, x$

A **function** or **map** from a set X to a set Y is a set of ordered pairs from X and Y (pairs like (x, y) are the coordinates in the graph of the function) satisfying the **uniqueness of image** property:

for all $x \in X$, there exists a **unique** $y \in Y$ that is related to the given x

Then we usually write $y = f(x)$ or $y = fx$, and make the sets involved clear by

$$f : X \to Y : x \mapsto f(x).$$

Note that a map is equivalent to its graph, as a set of ordered pairs of coordinates in $X - Y$ space; the graph of a map must not pass twice through any point in its domain—unlike a general relation.

A map $f : X \to Y$ may have any or none of the following properties:

injectivity (1 to 1) $\quad f(x) = f(y)$ implies $x = y$

surjectivity (onto)	im $f = Y$; denoted $f : X \twoheadrightarrow Y$
bijectivity (both)	injectivity and surjectivity

The **inclusion map** of a subset $A \subseteq X$ is the map

$$i : A \hookrightarrow X : a \mapsto a.$$

The **restriction** of a map $f : X \to Y$ to a subset of its domain $A \subseteq X$ is the composite map $f|_A = fi$.

The **Axiom of Choice** is required for a number of constructions in topology and a convenient form is this:

Every surjection has a right inverse.

That is, if $f : X \to Y$ is surjective, then we can always find a map $s : Y \to X$ such that $f \circ s = 1_Y$. Then s is called a **section** of f. Equivalently, given *any* collection (not necessarily countable) of sets, it is possible to choose one element from each.

Given a map

$$f : X \to Y : x \mapsto f(x)$$

we get free two maps on subsets, one going each way:

$$f_{\rightarrow} : \mathrm{Sub}X \to \mathrm{Sub}Y : A \mapsto \{f(x)|\ x \in A\}$$

$$f^{\leftarrow} : \mathrm{Sub}Y \to \mathrm{Sub}X : B \mapsto \{x \in X|\ f(x) \in B\}$$

Normally, we just use f to denote what is really $f_{\rightarrow}$, but it is important to be very careful in writing f^{-1} instead of $f^{\leftarrow}$, because whereas $f^{\leftarrow}$ always exists, f^{-1} may not.

1.2 Exercises

1. Give an example of a relation on $\mathbb{R}$ that is not a map.
2. Give an example of an equivalence relation on $\mathbb{Z}$.
3. Prove that an equivalence relation on a set S defines a **partition** of S (that is, separates S into disjoint nonempty subsets).
4. Find examples of maps for the following cases:
   (a) Surjective $f : \mathbb{R}^3 \to \mathbb{R}^2$.
   (b) Injective $f : \mathbb{R} \to \mathbb{R}^3$.
   (c) Bijective $f : \mathbb{R} \to \mathbb{R}$.
   (d) Bijective $f : [a, b] \to [c, d]$.
5. Precisely when does a map have an inverse?
6. If a map $f : \mathbb{R} \to \mathbb{R}$ is differentiable, what is a sufficient condition for it to be injective?
7. Prove that the inclusion map, $i : A \hookrightarrow X$, of a subset $A \subseteq X$ is the restriction of the identity map 1_X on X.

2 Categories and functors

Category theory provides some convenient and practical guidelines in mathematical constructions. It has particular relevance in algebraic topology because there we devise translations of difficult classification problems in topology into simpler ones in algebra, and category theory provides the accepted rules for such translations. A common way to prove that a mathematical space lacks a given property is to apply a functor and show that the image space lacks the corresponding property in the new category.

For our purposes, a category is a collection of sets with specified structures (group, topological, vector, ...) and the relevant structure-preserving maps (homomorphisms, continuous maps, linear maps, ...). Functors are the maps between categories which preserve the compositions of structure-preserving maps.

Now, each elementary composition of maps is a triangular diagram and often we can think of these triangles as the building blocks of categories, indeed, if you try to imagine a 'quantum of mathematics', it has to be something like a triangular diagram! Functors map the triangles in one category to those in another category; cofunctors do the same but reverse the direction of the arrows—recall the inverse image subset map $f^{\leftarrow}$ mentioned above which 'goes oppositely' to f itself.

2.1 Definitions

A **category** $\mathcal{C}$ is a directed graph, the vertices being called **objects** and the oriented edges **arrows** or **morphisms** of $\mathcal{C}$ (and these collections may be sets or proper classes) satisfying these axioms:

1. Every object A has an **identity** morphism $A \xrightarrow{1_A} A$.
2. Morphisms $A \xrightarrow{f} B$, $C \xrightarrow{g} D$ **compose** to give a morphism $A \xrightarrow{g \circ f} D$ if and only if $B = C$.
3. Composition is **associative**.
4. Identity morphisms always compose with any $A \xrightarrow{f} B$.

Intuitively, a category is a graph (or diagram) in which the essential structural elements are triangles reflecting the composition of arrows.

The idea of a **subcategory** is intuitively clear and illustrated by the inclusion $Ab \hookrightarrow Grp$, where Ab denotes the category of Abelian groups. A **functor** is a category-structure-preserving map between categories, so it needs to be declared on the elementary triangular diagrams and satisfying $F(1_A) = 1_{F(A)}$. If F had **reversed** the direction of the arrows in mapping triangles of $\mathcal{C}_1$ to those of $\mathcal{C}_2$, then we would call it a **cofunctor**. An example of the latter is the dualizing cofunctor of linear algebra. This sends every vector space A to the dual space A^* of real-valued linear maps on A and all linear maps $f : A \to B$ to the linear map

$$f^* : B^* \to A^* : \phi \mapsto \phi \circ f.$$

2.2 Exercises

1. Prove that the following are categories:
   (a) *Set*, with objects the class of all sets and morphisms all maps among them with map composition;
   (b) *Grp*, with objects the class of all groups and morphisms all group homomorphisms among them, with normal set theoretic composition;
   (c) $Vec_{\mathbb{R}}$, with objects the class of all real vector spaces and morphisms all $\mathbb{R}$-linear maps among them, with normal set-theoretic composition.
2. Investigate whether or not the relations:

   $$X \mapsto \mathrm{Sub}X, \quad \text{and} \quad f \mapsto f_{\rightarrow}$$

   $$X \mapsto \mathrm{Sub}X, \quad \text{and} \quad f \mapsto f_{\leftarrow}$$

   define a functor and a cofunctor, respectively.
3. The **forgetful functor** from *Grp* to *Set* simply forgets the group structure.
4. If $\mathcal{B}$ is a subcategory of $\mathcal{A}$, then it determines the **inclusion functor**.
5. Investigate for finite dimensional vector spaces the taking of double duals.

3 Groups

A **group** is the least structure in which we can define an internal operation which generalizes the multiplication and division on nonzero real numbers. A **field** is the nicest way in which two distinct groups can be fitted together so as to preserve the two identity elements as in the familiar example $(\mathbb{R}, +, \times)$ which is given by $\big((\mathbb{R}, +), (\mathbb{R} \setminus \{0\}, \times)\big)$. A **ring** is slightly weaker, lacking division, like $(\mathbb{Z}, +, \times)$.

A **vector space**, or **linear space**, is the nicest way in which a group (with a commutative operation $+$) can be combined with a field so as to preserve all three identity elements; a **module** is similar, but uses a ring instead of a field for its scalars.

For each of these, the appropriate maps which preserve the operations (hence all identities and inverses), between two structures of the same type, are called **homomorphisms**. Invertible homomorphisms are called **isomorphisms** and, for a given structure the set of self-isomorphisms or **automorphisms** forms a group. The fundamental concepts in group theory are enshrined in what we now call the category *Grp* of groups and group homomorphisms.

3.1 Definitions

A **group** is a set G together with a map

$$*: G \times G \;\rightarrow\; G : (g,h) \;\mapsto\; g * h,$$

called a **binary operation**, satisfying:

1. $*$ is associative: $(a*b)*c \;=\; a*(b*c)$ $(\forall\, a,b,c \in G)$;
2. $*$ has an identity element $e \in G$: $a * e = e * a = a$ $(\forall\, a \in G)$;
3. $*$ admits inverses: $(\forall\, a \in G)(\exists\, a^{-1} \in G):\; a * a^{-1} = a^{-1} * a = e.$

When there is no risk of confusion, we may omit the product symbol $*$ and write ab for $a * b$; however, it is quite common to be dealing with more than one group structure on the same set so care is needed. A group $(G,*)$ is called **abelian** or **commutative** if $a * b = b * a$ for all $a, b \in G$.

A map $\phi : G \to H$ between groups $(G,*), (H,\star)$ is a **group homomorphism** if it preserves the group operations:

$$\phi(a*b) \;=\; \phi(a) \star \phi(b) \qquad (\forall\, a,b \in G)\,.$$

If the homomorphism is from a group to itself then we call it an **endomorphism**.

A subset G' of a group G is a **subgroup** of G if the inclusion map $G' \overset{i}{\hookrightarrow} G$ is a group homomorphism; then G' is itself a group with the restriction of the operation of G. The **kernel** of a homomorphism $\phi : G \to H$ is the subgroup $\ker\phi = \phi^{\leftarrow}1_H$, where 1_H denotes the identity element of H.

If a group homomorphism $\phi : G \to H$ is bijective, then its inverse is also a group homomorphism, ϕ is called an **isomorphism** and the groups G and H are called **isomorphic**, written $G \cong H$. If an isomorphism is from a group to itself, then we call it an **automorphism**.

If H is a subgroup of G, then we define for each $g \in G$:

1. $gH = \{g * h \mid h \in H\}$, and $\{gH \mid g \in G\}$ the set of **right cosets** of H in G.
2. $Hg = \{h * g \mid h \in H\}$, and $\{Hg \mid g \in G\}$ the set of **left cosets** of H in G.

There is always a bijection between the sets of right and left cosets, but it may not be natural. When it is, we call the subgroup **normal** if $gH = Hg$ for all g. In this case the set of cosets itself forms a group, the **quotient group** denoted by G/H.

The number of elements in G is called the **order** of G, denoted $|G|$; if the order of a group is finite then we call it a finite group. If the smallest number of elements in a generating set is finite, then we call the group **finitely generated**. If $|G|$ is finite and G has a subgroup H, then H has a finite number of right cosets in G, called the **index** of H in G and denoted by $(G : H)$. It follows that, if $|G|$ is finite,

$$|G| = (G:H)\;|H|\,. \qquad \text{(Remember: } |G| \text{ finite!)}$$

This gives the famous theorem of Lagrange: if G is a *finite* group then the order of any subgroup divides the order of G. Hence groups of prime order have no nontrivial subgroups.

We can construct a group from a given set of elements by simple juxtaposition of the elements; the group consists of the set of all finite **words** made up from the given elements and their inverses, with composition of words by juxtaposition. This group is called the **free group** on the given elements. The free group on one generator is isomorphic to $(\mathbb{Z},+)$; a free group on more than one generator cannot be abelian. Many groups arise in practice as a set of generating elements together with some rules of combination. The **free product** $G * F$ of two groups consists of words made from both, with all internal products simplified in each.

The **direct product** $(G \times H, * \times \circ)$ of two groups $(G,*), (H,\circ)$ is the group defined on the product set $G \times H$ by

$$(g,h) * \times \circ\, (g',h') = (g * g', h \circ h')\,.$$

If two normal subgroups J, K of a group G can be found such that every $g \in G$ can be written uniquely in the form $g = jk$ for some $j \in J,\; k \in K$ and $J \cap K = \{e\}$, the trivial subgroup, then we say that G decomposes into the direct product of J and K.

The **commutator** of two elements a, b in a group G is the element $a^{-1}b^{-1}ab$. The subgroup $[G,G]$ of G generated by all of its commutators is called the **commutator subgroup**; the quotient group $G/[G,G]$ is always abelian. We call the process of taking this quotient **abelianizing** G.

3.2 Exercises

1. Show that the identity element in a group is unique, as are inverses.

2. The set $\{z \in \mathbb{C} \mid \; |z| = 1\}$, of unimodular complex numbers, forms an infinite group under multiplication. This is actually a topological group, homeomorphic to the unit circle.

3. The set of n^{th} roots of unity forms a group under multiplication.

4. Find all possible groups of orders 2, 3 and 4 by writing out possible entries in the matrix of products (the **group table**).

5. Given a finite group G and any set $a_1, a_2, \ldots, a_n$ of distinct elements from G, prove that these elements and their products among themselves and their inverses generate a subgroup of G.

6. The set of $n \times n$ nonsingular real matrices forms a group $GL(n, \mathbb{R})$, often just written $GL(n)$, the **general linear group**, under matrix multiplication. So does $O(n)$, the subset consisting of **orthogonal** matrices, and its subset $SO(n)$ consisting of those with determinant $+1$.

7. Prove that $GL(2)$ has a subgroup consisting of
$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \; \theta \in \mathbb{R} \right\}.$$
This is actually $SO(2)$, the **special orthogonal group** of 2×2 real matrices.

8. Find an isomorphism
$$f : SO(2) \to \{z \in \mathbb{C} | \; |z| = 1\}$$
and give its inverse.

9. Prove that, for all elements a in group G, the map
$$c_a : G \to G : x \mapsto a^{-1}xa$$
is an automorphism; find the inverse of c_a.

10. Prove that if we have a homomorphism $f : G \to H$ and H is abelian, then $\ker f$ contains all of the commutators in G.

11. If $\phi : G \longrightarrow H$ is a group homomorphism and e_H denotes the identity element in H, then:
    (a) $\ker\phi = \{g \in G \mid \phi(g) = e_H\}$ is a subgroup of G.
    (b) $\text{im }\phi = \{\phi(g) \in H \mid g \in G\}$ is a subgroup of H.

12. The set of n^{th} roots of unity forms a subgroup of the abelian group of unimodular complex numbers.

13. The map
$$\phi : \mathbb{Z} \longrightarrow \mathbb{S}^1 : k \longmapsto e^{ik2\pi}$$
is a group homomorphism from the additive group of integers $(\mathbb{Z}, +)$ to the multiplicative group of unimodular complex numbers.

14. $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$.

15. $GL(n; \mathbb{R})$ is not abelian if $n > 1$.

16. The **symmetric group** S_n of permutations of n objects is not abelian for $n > 2$.

17. Given groups G_1, G_2, show that the natural projections
$$p_i : G_1 \times G_2 \;\longrightarrow\; G_i : \;(g_1, g_2) \;\mapsto\; g_i \qquad (i = 1, 2)$$
are group homomorphisms from the direct product group.

18. If H, K are subgroups of G then $H \cap K$ is also a subgroup, but $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$. If H, K are normal subgroups then so is HK.

19. If G has no nontrivial subgroups (that is, only $\{e\}$ and G are subgroups of G) then G is generated by one element (so G is called a **cyclic** group) and has prime order.

4 Topology

A topological space is a set with the least structure necessary to define the concepts of nearness and continuity; you have met examples in real and complex analysis and perhaps also as a metric space (a set with the least structure necessary to support the concept of distance).

General topology is concerned with the study of topological spaces and maps among them while algebraic topology is concerned with the casting of topological problems into easier algebraic form using functors.

4.1 Definitions

A **metric space** (X, d) is a nonempty set X and a map $d : X \times X \to \mathbb{R}$, called a **metric** or **Hausdorff distance function**, satisfying the natural requirements of a distance function:

1. $d(x, y) = d(y, x) \quad \forall x, y \in X$ (Symmetry)
2. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$ (Positive definiteness)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

The standard metric on a normed vector space is simply the norm of the difference between two points. In geometry, **Euclidean n-space**, $\mathbb{E}^n$, is the metric space of points in $\mathbb{R}^n$ with distance function

$$d(p, q) = ||q - p||.$$

A **topological space** $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T}$ of subsets, so $\mathcal{T} \subseteq P(X)$, satisfying:

1. $\varnothing, X \in \mathcal{T}$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

We call $\mathcal{T}$ **the topology** of the space $(X, \mathcal{T})$ or **a topology** on the set X. Elements of $\mathcal{T}$ are called **open sets** in the topological space $(X, \mathcal{T})$ or they are called **$\mathcal{T}$-open sets** of X. A **base** for a topology $\mathcal{T}$ on X is a collection $\mathcal{B} \subseteq \mathcal{T}$ of open sets of X such that every member of $\mathcal{T}$ is expressible as a union of members of $\mathcal{B}$.

Every metric space (X, d) has a topology $\mathcal{T}_d$ determined by d. A subset A of X is d-open in (X, d) if it contains an open ball around each of its points, and we define $\mathcal{T}_d$ to be the set of d-open subsets. So a base for a metric topology is the collection of all open balls.

Let $(X, \mathcal{T})$ be a topological space. A set A is **closed** in $(X, \mathcal{T})$ if $X \setminus A$ is open in $(X, \mathcal{T})$ (that is, closed if it is the complement of an open set). Sometimes $X \setminus A$ is denoted $X - A$. A point $x \in X$ is a **limit point** of $A \subseteq X$ in $(X, \mathcal{T})$ if every neighborhood of x meets $A \setminus \{x\}$ non-emptily. A limit point of A need not be in A, but it turns out that A is closed in $(X, \mathcal{T})$ if and only if A contains all of its limit points.

The **closure** $\bar{A}$ of a set A in a topological space is the union of A with all of its limit points; that is, the smallest closed set containing A. The **interior**, int A (also denoted $(A)^\circ$, when convenient) of A is the largest open set contained in A. A is **dense in** $(X, \mathcal{T})$ if $\bar{A} = X$. The **boundary** or **frontier** of a set A is $\partial A = \bar{A} \cap \overline{X \setminus A}$.

A map between topological spaces $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is called **continuous** if

$$\forall B \in \mathcal{T}', \ f^{\leftarrow} B \in \mathcal{T}.$$

A continuous map f is called
open if $U \in \mathcal{T} \ \Rightarrow \ fU \in \mathcal{T}'$
and is called
closed if $U \in \mathcal{T} \ \Rightarrow \ f(X \setminus U)$ is closed in Y.
Open, closed, and continuous are independent properties.
A map $f : X \to Y$ is called a **homeomorphism** if

$$f \text{ is continuous}, \quad f^{-1} \text{ exists, and} \quad f^{-1} \text{ is continuous;}$$

that is, if f is **bijective and bicontinuous**; then we say that X and Y are **homeomorphic** and write $X \cong Y$.

For proofs in topology we usually juggle the properties of open and closed sets. We can get open sets in relatively few ways:

1. directly from $\mathcal{T}$;
2. as complements of closed sets;
3. as inverse images of open sets by continuous maps.

Given a set X, there is a **partial order** $\leq$ defined on the set of all topologies available on X:

$$\begin{aligned} \mathcal{T}_1 \leq \mathcal{T}_2 \quad &\Longleftrightarrow \quad A \in \mathcal{T}_1 \Longrightarrow A \in \mathcal{T}_2 \\ &\Longleftrightarrow \quad A \text{ is } \mathcal{T}_1\text{-open} \Longrightarrow A \text{ is } \mathcal{T}_2\text{-open.} \end{aligned}$$

For $\mathcal{T}_1 \leq \mathcal{T}_2$ we often write $\mathcal{T}_2 \geq \mathcal{T}_1$. A set with a partial order (sometimes called a pre-order) is called a **poset**.

In the case that $\sim$ is an equivalence relation on a topological space X, the set of equivalence classes is $Y = X/\!\sim$ and we use the largest topology $\mathcal{T}_\sim$ on Y where

$$\mathcal{T}_\sim = \sup\{\text{topologies on } Y \text{ making } \pi : X \twoheadrightarrow Y : x \mapsto [x]_\sim \text{ continuous}\}.$$

The following are easily deduced. Let $(X, \mathcal{T})$ be a topological space and, for some set Y, suppose that we have either:

1. a map $f : Y \longrightarrow X$, or
2. a map $g : X \longrightarrow Y$.

Then in the first case,

$$f^{\leftarrow}\mathcal{T} \;=\; \{f^{\leftarrow}G \mid G \in \mathcal{T}\}$$

is the *smallest* topology on Y that makes $(Y, f^{\leftarrow}\mathcal{T}) \xrightarrow{f} (X, \mathcal{T})$ continuous. In the second,

$$g\mathcal{T} = \{H \subseteq Y \mid g^{\leftarrow}H \in \mathcal{T}\} \cup \{Y\}$$

is the *largest* topology on Y that makes $(X, \mathcal{T}) \xrightarrow{g} (Y, g\mathcal{T})$ continuous. We call $f^{\leftarrow}\mathcal{T}$ the **coinduced** or **inverse image** topology on Y, and $g\mathcal{T}$ the **induced** or **quotient** topology on Y. The prefix **'co-'** here means that a map is going backwards.

There are two special cases of induced topologies that occur frequently:

Inclusions For $Y \subseteq X$, the inclusion map $i : Y \hookrightarrow X$ determines $i^{\leftarrow}\mathcal{T} = \mathcal{T}|_Y$ the **subspace topology** on Y.

For example, $i : \mathbb{S}^1 \hookrightarrow \mathbb{B}^2 : x \mapsto x$ where we regard the unit circle as the boundary subset, $\partial\mathbb{B}^2$, of the unit disk $\mathbb{B}^2$:

$$\mathbb{S}^1 \cong \partial\mathbb{B}^2 \subseteq \mathbb{B}^2 \subseteq \mathbb{E}^2.$$

The topology $\mathcal{T}$ on $\mathbb{B}^2$ is the standard metric topology with base consisting of open disks like $\operatorname{int} B(x, r)$. The subspace topology on the boundary $\mathcal{T}|_{\partial\mathbb{B}^2}$ consists of those $A \cap \partial\mathbb{B}^2$ where $A \in \mathcal{T}$. In general, $\mathcal{T}|_Y = \{A \cap Y \mid A \in \mathcal{T}\}$: we want the *smallest* topology (that co-induced by inclusion); we don't want to introduce spurious open sets.

Partitions Here we have $\pi : X \twoheadrightarrow Y : x \mapsto [x]_\sim$ under some equivalence relation $\sim$ on X. B is open in $Y = X/\!\sim$ if and only if $\pi^{\leftarrow}B$ is open in X. We want the *largest* topology here, to get the maximum influence of $(X, \mathcal{T})$. This is called the **quotient** or **identification topology** induced by π.

We *always* use these topologies unless stated otherwise. When we have to arrange continuity of two or more maps in the construction of a topology, we make use of the fact that every partially ordered set has unique supremum and infimum elements, and use the **sup** and **inf** topologies so defined on the set of all topologies.

Consider the product set $Y = \prod_{\alpha \in A} X_\alpha$ with projections

$$p_\beta : \prod_{\alpha \in A} X_\alpha \twoheadrightarrow X_\beta : (X_\alpha)_{\alpha \in A} \mapsto \; X_\beta\,.$$

The **product topology** on the product set is

$$\sup\{p_\alpha \mathcal{T}_\alpha \mid \alpha \in A\}.$$

(Used when combining lines to form a plane to represent figures in geometry).
Consider the disjoint union set $Y = \coprod_{\alpha \in A} X_\alpha$ with injection maps

$$f_\beta : X_\beta \longrightarrow \coprod_{\alpha \in A} X_\alpha : x \mapsto x\,.$$

The **coproduct topology** on disjoint unions is

$$\inf\{f_\alpha \mathcal{T}_\alpha \mid \alpha \in A\}.$$

The inclusion map, $i : A \hookrightarrow X$, of a subset $A \subseteq X$ is the restriction of the identity map 1_X on X, and hence it is always continuous. This means that every subset of a topological space is automatically a sub-topological space, unlike the corresponding situation in algebra. There, for example, only some inclusion maps are group homomorphisms, namely those which have subgroups as their domains.

4.2 Exercises

1. Give continuous and discontinuous examples of maps that are
   (a) surjective
   (b) injective
   (c) bijective
2. Give examples of maps $f, g : \mathbb{R} \to \mathbb{R}$, with $g \circ f$ continuous and f continuous, but g discontinuous.
3. Prove that as a subspace of $\mathbb{R}^{2\times 2} = \mathbb{R}^4$, $SO(2)$ is homeomorphic to $\mathbb{S}^1$, the unit circle in the complex plane. Deduce that the standard group structures on these two spaces are isomorphic and hence we have an isomorphism of topological groups.
4. Find a continuous invertible map
   $$f : [0, 2\pi) \to \mathbb{S}^1$$
   so f^{-1} exists, but such that f^{-1} is *not* continuous.
5. Find a homeomorphism (called **stereographic projection**)
   $$f : \mathbb{S}^2 \setminus \{\text{North Pole}\} \to \mathbb{R}^2,$$
   by considering $\mathbb{R}^2$ as a plane through the equator and all lines from the North Pole to this plane. [Hint: Try the case for $\mathbb{S}^1$ first.]

5 Group actions

In algebra, geometry and topology we often exploit the fact that important structures arise from families of morphisms that are indexed by a group. For example, rotations in the plane about the origin are indexed by the unimodular group of complex numbers; we say that this group **acts on the plane** and the **orbit** of a point at distance r from the origin is the circle of radius r.

5.1 Definitions

A group G is said to **act** on a set (for example, a group, vector space, manifold, topological space) X **on the left** if there is a map (for example, homomorphism, linear, smooth, continuous)

$$\alpha : G \times X \longrightarrow X : (g, x) \longmapsto \alpha_g(x)$$

such that $\alpha_{g*h}(x) = \alpha_g(\alpha_h(x))$ and $\alpha_e(x) = x$ for all $x \in X$. Normally, we shall want each $\alpha_g : X \to X$ to be an isomorphism in the category for X; in this case, an **action** is the same as a representation of G *in* the automorphism group of X, or a representation *on* X. We sometimes abbreviate the notation to

$g \cdot x$, especially when α is fixed for the duration of a discussion. There is a dual theory of actions on the right; we have to keep the concepts separate because every group acts on itself by its group operation, but it may be different on the right from on the left.

The **orbit** of $x \in X$ under the action α of G is the set

$$G \cdot x = \{\alpha_g(x) \mid g \in G\}.$$

It is easy to show that the orbits partition X, so they define an equivalence relation on X:

$$x \sim y \iff \exists\, g \in G \text{ with } \alpha_g(x) = y.$$

The quotient object (set, space, *etc.*) is called the **orbit space** and denoted by X/G.

The **stabilizer** or **isotropy subgroup** of x is defined to be the set

$$\mathrm{stab}_G(x) = \{g \in G \mid \alpha_g(x) = x\},$$

and it is always a subgroup of G.

The action is called **transitive** if for all $x, y \in X$ we can find $g \in G$ such that

$$\alpha_g(x) = y \qquad (\text{so also } \alpha_{g^{-1}}(y) = x),$$

free if the only α_g with a fixed point has $g = e$ (the identity of G), and **effective** if

$$\alpha_g(x) = x \quad (\forall\, x \in X) \implies g = e.$$

Note that an action being transitive is equivalent to it having exactly one orbit, or to its orbit space being a singleton.

The situations of most practical interest are when:

- X is a group or vector space—especially $\mathbb{R}^n$;
- G, X are **topological groups**, so each has a topology with respect to which its binary operation and the taking of inverses is continuous;
- G is a topological group and X is a topological space;
- G is a **Lie group**, so G has a differentiable structure with respect to which its binary operation and the taking of inverses is smooth, and X is a smooth manifold. Here smooth means all derivatives of all orders exist and are continuous. Important examples of Lie groups are $\mathbb{R}^n$, $GL(n)$ and $\mathbb{S}^1$, where the differentiability arises from that of the underlying real functions.

5.2 Examples

1. Find a group G consisting of four, 2×2 real matrices such that G acts on the plane $\mathbb{R}^2$.
2. The group $SO(2)$ of rotations in a plane acts on a sphere $\mathbb{S}^2$ as rotations of angles of longitude. The orbits are circles of latitude and the quotient space by this action is the interval $[-1, 1]$. The action is neither transitive nor free, but it is effective.
3. Prove that $SO(2)$ defines a left action on $\mathbb{R}^2$ by

   $$\rho : SO(2) \times \mathbb{R}^2 \to \mathbb{R}^2 : (A, p) \mapsto L_A p$$

   where $L_A p$ denotes matrix multiplication of the coordinate column vector p by the matrix A. To establish this you need to show that the map ρ is well-defined and that it satisfies two rules for all $p \in \mathbb{E}^2$ and all $A, B \in SO(2)$, namely

   Product $L_A(L_B p) = L_{AB} p$

   Identity $L_I p = p$

   [In fact, the whole of the general linear group $GL(2)$ acts on $\mathbb{R}^2$.]
4. Prove that the action ρ is effective but neither free nor transitive. Find the orbits under this action of the points on the x-axis of $\mathbb{R}^2$.

5. Prove that the action ρ preserves the scalar product; that is, for all $p, q \in \mathbb{R}^2$ and all $A \in SO(2)$,

$$L_A p \cdot L_A q = p \cdot q$$

Hence deduce that the action preserves Euclidean angles, lengths and areas.

6. Show that

$$L_J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO(2)$$

and find the image under L_J of the unit square in the upper right quadrant of $\mathbb{R}^2$. [Hint: Check the edge vectors.] Find an element $K \in GL(2)$ with $K \notin SO(2)$ and $\det K = -1$. This defines a linear map L_K; compare its effect on the unit square with the image found for L_J.

6 Euclidean space

We formalize our intuitive notions of ordinary space, subject to Euclid's axioms. In this course we shall be concerned with three dimensional $\mathbb{E}^3$ but the basic definitions of points, difference vectors and distances are the same for all $\mathbb{E}^n$ with $n = 1, 2, 3, \ldots$; of course, in dimensions higher than 3, the extra directions will arise from other features than ordinary space—such as time, temperature, pressure etc. The important fact to hang onto is that $\mathbb{E}^3$ consists of points represented by coordinates $p = (p_1, p_2, p_3)$ while the directed difference between a pair of such points p, q is a vector $\overline{q - p}$ with components $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$. In modern mathematics, it is customary to omit the overbar when writing vectors and this will be our usual practice; we identify vectors with their sets of components and points with their sets of coordinates. Often, we use the coordinates (x, y, z) for points in $\mathbb{E}^3$ and denote by $\mathbb{E}^2$ the set of points in $\mathbb{E}^3$ with $z = 0$; then we abbreviate $(x, y, 0)$ to (x, y).

The first objective of this course is to develop the geometry of curves and surfaces in $\mathbb{E}^3$. The basic ideas are very simple: a curve is a continuous image of an interval and a surface is a continuous image of a product of intervals; in each case the intervals may be open or closed or neither.

6.1 Definitions

In geometry it is convenient and helpful to distinguish between $\mathbb{R}^n$, the real **vector space** of n-tuples of real numbers, and Euclidean space $\mathbb{E}^n$, the **point space** of n-tuples of real numbers. Euclidean space is equipped with the **difference map**:

$$\text{diff} : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}^n : (p, q) \mapsto q - p$$

and the **distance map**:

$$\text{dist} : \mathbb{E}^n \times \mathbb{E}^n \to [0, \infty) : (p, q) \mapsto ||q - p||.$$

here, $||\ ||$ denotes the operation of taking the **norm** or absolute value of the vector, defined by

$$||(q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n)|| = +\sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + \ldots + (q_n - p_n)^2}$$

Note that the norm of a vector is always a non-negative real number.

Thus, we view $\mathbb{E}^n$ as the set of points, together with the standard Euclidean angles and Pythagorean distances; then $\mathbb{R}^n$ provides the vectors of **directed differences** between ordered pairs of points. Not all books make this distinction so you need to be prepared to encounter the unstated identification $\mathbb{E}^n = \mathbb{R}^n$ and beware of confusing vectors and points.

The **standard unit sphere** $\mathbb{S}^n$ in a Euclidean n-space is the set of points unit distance from the origin; we shall often use $\mathbb{S}^1$ in $\mathbb{E}^2$ and $\mathbb{S}^2$ in $\mathbb{E}^3$.

Given a point $p \in \mathbb{E}^3$ and a vector $v \in \mathbb{R}^3$, there is always a unique point $q \in \mathbb{E}^3$ such that $(q - p) = v$; intuitively, q is at the point of the arrow v when its tail is at p. Then we can write $q = p + v$ and the **line segment** from p (in the direction v) to q is given in parametric form by

$$L : [0, 1] \to \mathbb{E}^3 : t \mapsto p + tv$$

or equivalently

$$L : [0, 1] \to \mathbb{E}^3 : t \mapsto (p_1, p_2, p_3) + t(v_1, v_2, v_3)$$

and we say that the **tangent vector**, or **velocity vector** of this line at $t \in [0, 1]$ is v. Thus, we can write this as the derivative, which is the limit of differences:

$$D_t L = \lim_{h \to 0} \frac{L(t+h) - L(t)}{h} = \frac{dL}{dt} = v$$

because L is a linear function of t, since p and v are constant.

As we shall see, a curve may have a nonlinear dependence on its parameter and a velocity vector that varies in magnitude and direction, so curves are natural generalisations of line segments.

A **curve** in $\mathbb{E}^3$ with parameter t satisfying $a \leq t \leq b$ is a continuous map

$$\alpha : [a, b] \to \mathbb{E}^3 : t \mapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \qquad (*)$$

Note that α is a map and its image, (also called track or path), $\alpha([a, b])$, is a subset of $\mathbb{E}^3$; we keep these concepts distinct. The curve starts at the point $\alpha(a)$ and ends at the point $\alpha(b)$. Sometimes, one or both endpoints of the curve are absent, so in general the domain of a curve may be any kind of interval.

For our purposes, we shall suppose that our curves are **differentiable**, in the sense that the components, $\alpha_1, \alpha_2, \alpha_3$, are real functions of t possessing derivatives of all orders—so no corners like those in the graph of $|x|$. The **tangent vector** or **velocity** of α is the vector valued map $D_t \alpha = \alpha'$ which in components is given by

$$\alpha' : [a, b] \to \mathbb{R}^3 : t \mapsto (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$$

and its **speed** is the absolute value of the velocity vector. The **acceleration** of α is the vector $\alpha'' = (\alpha')'$, given by the derivative of the velocity. Observe that the velocity and acceleration vectors are attached to the curve and change as the parameter moves the point of attachment. Occasionally, we may need to consider continuous curves that fail to have a tangent vector at isolated points—for example they may have a corner.

We are particularly interested in **regular** curves which are differentiable and have nowhere zero velocity (they are always going somewhere, not stopped); then we usually choose the parameter set $0 < s < L$ to make α' a unit vector for all s and L turns out to be the total length of the curve.

The **length** of the curve (*) is defined as the integral of the absolute value of the velocity over the domain $[a, b]$

$$\text{Length}[\alpha] = \int_{[a,b]} \|\alpha'(t)\| = \int_a^b \sqrt{\alpha_1'(t)^2 + \alpha_2'(t)^2 + \alpha_3'(t)^2}\, dt$$

and it is independent of reparametrization. Unit speed curves are parametrized by arc length because then $\|\alpha'(t)\| = 1$ for all t.

The **Euclidean group** $E(n)$ consists of all isometries of Euclidean n-space $\mathbb{E}^n$. Isometries can always be written as an ordered pair from $O(n) \times \mathbb{R}^n$ with action on $\mathbb{E}^n$ given by

$$(O(n) \times \mathbb{R}^n) \times \mathbb{E}^n \longrightarrow \mathbb{E}^n : ((\alpha, u), x) \longmapsto \alpha(x) + u$$

and composition

$$(\alpha, u)(\beta, v) = (\alpha\beta, \alpha(v) + u)\,.$$

Thus, topologically $E(n)$ is the product $O(n) \times \mathbb{R}^n$ but algebraically it is not the product group. It is called a **semidirect product** of $O(n)$ and $\mathbb{R}^n$. For the case $n = 2$ find discrete subgroups $G < E(2)$ such that $\mathbb{R}^n/G$ is: (i) the cylinder; (ii) the torus.

6.1.1 Exercises

1. Show that a parametric equation for the line segment from $(3, 5)$ to $(6, 1) \in \mathbb{E}^2$ is given by

   $$\ell : [0, 1] \to \mathbb{E}^2 : t \mapsto (3, 5) + t(3, -4).$$

   Note that the line segment is the **image** of the map ℓ. What is the tangent vector of ℓ?

2. Find another parametric equation for the line segment in the previous example, but having a tangent vector with half the magnitude of that used for ℓ.

3. Clearly, there is a well-defined continuous curve given by

   $$h : [-1, 1] \to \mathbb{E}^2 : t \mapsto (t, |t|)$$

   but what happens to its tangent vector at $t = 0$? Does the curve have a length?

4. Express the equation for the parabola $y = 3x^2 + 2$ in parametric form and find its velocity, speed and acceleration. Show that the same image can be obtained by another curve with velocity vector the negative of that of the first one.

5. Verify that a parametric equation for the curve defining the unit circle in $\mathbb{E}^2$ with centre at the origin O
$$\mathbb{S}^1 = \{p \in \mathbb{E}^2 | \ \mathrm{dist}(p, O) = 1\}$$
is given by
$$f : [0, 2\pi] \to \mathbb{E}^2 : t \mapsto (\cos t, \sin t).$$
Find its velocity, speed, acceleration and length. Show that the same image can be obtained by another curve with velocity vector the negative of that of the first one.

6. Investigate the hyperbola
$$f : [-1, 1] \to \mathbb{E}^2 : t \mapsto (\cosh t, \sinh t).$$

7. Plot the limaçon (French name for slug—why?) given by
$$l : [0, 2\pi] \to \mathbb{E}^2 : t \mapsto ((2\cos t + 1)\cos t, (2\cos t + 1)\sin t)$$
and show that the curve passes twice through the origin in different directions, which emphasises why we have to specify the parameter value and not the point on the curve when we require the velocity vector.

8. Verify that a parametric equation for the unit 2-sphere $\mathbb{S}^2$ in $\mathbb{E}^3$ is given by
$$g : [0, 2\pi] \times [-\pi/2, \pi/2] :\to \mathbb{E}^3 : (u, v) \mapsto (\cos v \cos u, \cos v \sin u, \sin v).$$
Find a parametric equation for the equator of this sphere, and for a perpendicular circle of longitude.

9. Find a parametric equation for a sphere of radius a.

10. Find a parametric equation for an ellipsoid with lengths of its principal semi-axes having values a, b, c.

11. It is clear that $GL(3)$, which acts on $\mathbb{E}^3$, has a subgroup $SO(3)$, consisting of 3×3 real matrices having determinant $+1$. Find three distinct subgroups of $SO(3)$, consisting of rotations around the three coordinate axes, respectively, by finding three group homomorphisms $SO(2) \to SO(3)$ with trivial kernels.

12. Use the subgroups of $SO(3)$ found in the previous exercise, and the parametric equation for the equator of $\mathbb{S}^2$ found above, to show how any other great circle on $\mathbb{S}^2$ can be found by appropriate combinations of rotations of the equator.

7 Regular Surfaces

The notion of a surface is a generalization of that of a curve: a surface is a continuous image of a *product* of (any kinds of) intervals; in practice, we make surfaces like a patchwork quilt, from a collection of overlapping pieces. Recall that in our study of curves we concentrated on **regular** curves—which had nowhere zero derivatives for component functions. There are three parts to this: existence of the continuous curve, existence of its tangent vector at all points, nowhere zero speed. This meant that, locally, the curve was homeomorphic to open subintervals of its domain and had a well-defined tangent vector at all parameter values so no corners (tangent vectors are undefined at corners as we see in the graph of $|x|$) but globally the curve may have had self-intersections.

In this course we shall study the corresponding generalization to define regular surfaces such that locally they are homeomorphic to open sets (called **coordinate patches** or just **patches**) in $\mathbb{E}^2$ and with a well-defined tangent plane at every point (the apex of a cone does not have a well-defined tangent plane). To achieve this, we need the maps defining our surface to be injective and with rank 2 Jacobians on the patches; so our patches are not allowed to generate self-intersections in their images. This means that the 2×2 matrix of partial derivatives on the patch must have nonzero determinant.

7.1 Computing

The computer algebra package Mathematica [10] can be used very effectively to perform calculus and create graphics for curves and surfaces; Gray [6] provides the necessary Mathematica input to plot and study virtually all named curves and surfaces and perform analytic calculations on them—including the solution of geodesic equations on the surfaces and construction of curves with prescribed curvature and torsion. The necessary files can be found via the web server
`http://www.ma.umist.ac.uk/kd/mmaprogs/AREADMEFILE`
An important source of curves is from solutions of ordinary differential equations. Gray et al. [7] have provided a definitive text on this subject, together with associated Mathematica functions to solve ordinary differential equations analytically and numerically, and plot families of solutions. The necessary files can be found via the web server
`http://www.ma.umist.ac.uk/kd/ode/ode.htm`

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