Matrices Example Sheet I Solutions

We recall the matrices:

$$A = \begin{pmatrix} -1 & 0 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & -2 \end{pmatrix}$$

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$$D = \begin{pmatrix} 4 & 0 & 3 \\ \frac{1}{2} & -2 & 3 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ 2 & 3 & -1 \\ 0 & 7 & 0 \end{pmatrix}$$

1.(i) We have

$$A + B = \begin{pmatrix} -1 & 0 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

which cannot be done as the matrices don't have the same size.

1.(ii) We have

$$A + E = \begin{pmatrix} -1 & 0 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -2 & 0 \\ 2 & 3 & -3 \end{pmatrix}$$

1.(iii) We have

$$C + F = \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & \text{-2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & \text{-2} & 0 \\ 2 & 3 & \text{-1} \\ 0 & 7 & 0 \end{pmatrix}$$

which cannot be done as the matrices don't have the same size.

1.(iv) We have

$$D + D = \begin{pmatrix} 4 & 0 & 3 \\ \frac{1}{2} & \text{-2} & 3 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 3 \\ \frac{1}{2} & \text{-2} & 3 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 6 \\ 1 & \text{-4} & 6 \end{pmatrix}$$

2.(i) We have

$$AB = \begin{pmatrix} \text{-1} & 0 & 5 \\ 2 & \text{-1} & 0 \\ 0 & 2 & \text{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+0+10 \\ 2-3+0 \\ 0+6-2 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 4 \end{pmatrix}$$

2.(ii) We have

$$BA = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} \text{-1} & 0 & 5 \\ 2 & \text{-1} & 0 \\ 0 & 2 & \text{-1} \end{pmatrix}$$

which cannot be done as the number of columns of B is not equal to the number of rows of A.

2.(iii) We have

$$CD = \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & \text{-2} \end{pmatrix} \begin{pmatrix} 4 & 0 & 3 \\ \frac{1}{2} & \text{-2} & 3 \end{pmatrix} = \begin{pmatrix} 8+\frac{1}{2} & 0-2 & 6+3 \\ 0+0 & 0+0 & 0+0 \\ 4-1 & 0+4 & 3-6 \end{pmatrix} = \begin{pmatrix} \frac{17}{2} & -2 & 9 \\ 0 & 0 & 0 \\ 3 & 4 & -3 \end{pmatrix}$$

2.(iv) We have

$$EF = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ 2 & 3 & -1 \\ 0 & 7 & 0 \end{pmatrix}$$

which cannot be done as the number of columns of E is not equal to the number of rows of F.

3.(i) We have

$$4D = 4 \begin{pmatrix} 4 & 0 & 3 \\ \frac{1}{2} & -2 & 3 \end{pmatrix} = \begin{pmatrix} 16 & 0 & 12 \\ 2 & -8 & 12 \end{pmatrix}$$

3.(ii) We calculated $A + E$ in question **1.(ii)**, so we just need to do the multiplication:

$$(A+E)B = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -2 & 0 \\ 2 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0+3+10 \\ 3-6+0 \\ 2+9-6 \end{pmatrix} = \begin{pmatrix} 13 \\ -3 \\ 5 \end{pmatrix}$$

3.(iii) We have

$$EB = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+3+0 \\ 1-3+0 \\ 2+3-4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

3.(iv) We calculated AB and EB in previous questions, so we just add them:

$$AB + EB = \begin{pmatrix} 9 \\ -1 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ -3 \\ 5 \end{pmatrix}$$

(Note that this is the same as $(A+E)B$ — this is an instance of the distributive law.)

3.(v) We have

$$AE = \begin{pmatrix} -1 & 0 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -1+0+10 & -1+0+5 & 0+0-10 \\ 2-1+0 & 2+1+0 & 0+0+0 \\ 0+2-2 & 0-2-1 & 0+0+2 \end{pmatrix} = \begin{pmatrix} 9 & 4 & -10 \\ 1 & 3 & 0 \\ 0 & -3 & 2 \end{pmatrix}$$

3.(vi) We just calculated AE, so now just multiply on the right by C:

$$(AE)C = \begin{pmatrix} 9 & 4 & -10 \\ 1 & 3 & 0 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 18+0-10 & 9+0+20 \\ 2+0+0 & 1+0+0 \\ 0+0+2 & 0+0-4 \end{pmatrix} = \begin{pmatrix} 8 & 29 \\ 2 & 1 \\ 2 & -4 \end{pmatrix}$$

3.(vii) We have

$$EC = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2+0+0 & 1+0+0 \\ 2+0+0 & 1+0+0 \\ 4+0-2 & 2+0+4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 6 \end{pmatrix}$$

3.(viii) We just calculated EC, so now we just need to multiply on the left by A:

$$A(EC) = \begin{pmatrix} -1 & 0 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} -2+0+10 & -1+0+30 \\ 4-2+0 & 2-1+0 \\ 0+4-2 & 0+2-6 \end{pmatrix} = \begin{pmatrix} 8 & 29 \\ 2 & 1 \\ 2 & -4 \end{pmatrix}$$

(Note that this is the same as $(AE)C$ — an instance of the associative law of matrix multiplication.)

Matrices Example Sheet II Solutions

1. We want to calculate $\det A = \begin{vmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 2 & -3 & 3 \end{vmatrix}$. We could expand using any row or column, but it's normal to use the top row — we just have to bear in mind the pattern: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ Now we have:

$$|A| = 1\begin{vmatrix} 1 & 3 \\ -3 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 1(3+9) + 2(0-6) - 1(0-2) = 12 - 12 + 2 = 2$$

and so the determinant is 2.

2. Since the determinant is not zero, we can work out the inverse by using minors. Recall that each minor m_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and the j^{th} column of the original matrix. Thus:

$$m_{11} = \begin{vmatrix} 1 & 3 \\ -3 & 3 \end{vmatrix} = 12 \qquad m_{12} = \begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} = -6 \qquad m_{13} = \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} = -2$$

$$m_{21} = \begin{vmatrix} -2 & -1 \\ -3 & 3 \end{vmatrix} = -9 \qquad m_{22} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \qquad m_{23} = \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} = 1$$

$$m_{31} = \begin{vmatrix} -2 & -1 \\ 1 & 3 \end{vmatrix} = -5 \qquad m_{32} = \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = 3 \qquad m_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$$

Thus, the matrix of minors is

$$\begin{pmatrix} 12 & -6 & -2 \\ -9 & 5 & 1 \\ -5 & 3 & 1 \end{pmatrix}$$

Now, we have to take the transpose of this (i.e. flip rows and columns), and put in minus signs according to the chessboard pattern:

$$\begin{pmatrix} + & \text{-} & + \\ \text{-} & + & \text{-} \\ + & \text{-} & + \end{pmatrix}$$

and finally divide by the determinant, which is 2. Thus

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 12 & 9 & \text{-}5 \\ 6 & 5 & \text{-}3 \\ \text{-}2 & \text{-}1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & \frac{9}{2} & -\frac{5}{2} \\ 3 & \frac{5}{2} & -\frac{3}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

It's a good idea to check the result:

$$A\,A^{-1} = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 2 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 6 & \frac{9}{2} & -\frac{5}{2} \\ 3 & \frac{5}{2} & -\frac{3}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 6-6+1 & \frac{9}{2}-5+\frac{1}{2} & -\frac{5}{2}+3-\frac{1}{2} \\ 0+3-3 & 0+\frac{5}{2}-\frac{3}{2} & -\frac{3}{2}+\frac{3}{2} \\ 12-9-3 & 9-\frac{15}{2}-\frac{3}{2} & -5+\frac{9}{2}+\frac{3}{2} \end{pmatrix}$$

as required.

3. The simultaneous equations

$$\begin{array}{rcrcrcr} x & - & 2y & - & z & = & 2 \\ & & y & + & 3z & = & 0 \\ 2x & - & 3y & + & 3z & = & -4 \end{array}$$

can be written in matrix form:

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 2 & -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

Now, we see that the matrix on the left is A, as in questions **1** and **2**. So let's multiply on the left of both sides by A^{-1}, so as to cancel off A:

$$\begin{pmatrix} 6 & \frac{9}{2} & -\frac{5}{2} \\ 3 & \frac{5}{2} & -\frac{3}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 2 & -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 & \frac{9}{2} & -\frac{5}{2} \\ 3 & \frac{5}{2} & -\frac{3}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 & \frac{9}{2} & -\frac{5}{2} \\ 3 & \frac{5}{2} & -\frac{3}{2} \\ -1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

now, doing the multiplication:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 + 0 + \frac{20}{2} \\ 6 + 0 + \frac{12}{2} \\ -2 + 0 - \frac{4}{2} \end{pmatrix} = \begin{pmatrix} 22 \\ 12 \\ -4 \end{pmatrix}$$

and so $x = 22$, $y = 12$ and $z = -4$ is the solution.