

2M1 Maths Tutorial: Series

1. Find a general expression u_r for the r^{th} term in the following infinite series (starting with $r = 1$), and check for convergence.

- (a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$
- (b) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$
- (c) $0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3} + \dots$
- (d) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
- (e) $1 + \frac{2}{x} + \frac{6}{x^2} + \frac{24}{x^3} + \dots$
- (f) $x + \frac{x^2}{2^x} + \frac{x^3}{3^x} + \frac{x^4}{4^x} + \dots$
- (g) $\frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{5}{32} + \frac{3}{32} + \frac{7}{128} + \dots$
- (h) $2 + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \dots$
- (i) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

2. Find the MacLaurin series for $(1+x)^4$, and show that this is the same as its binomial expansion.

3. Find the first three non-zero terms of the MacLaurin series for the following:

- (a) $\cos^2 x$
- (b) $x \ln(x+1)$
- (c) e^{-x^2}

4. Given $f(x) = \ln x$:

- (a) Integrate $f(x)$ then expand the result as a 3^{rd} order Taylor polynomial about $x = 1$.
- (b) Expand $f(x)$ about $x = 1$, integrate the result, showing this to be consistent (to third order) with the answer to part (a).

5. (a) Find the Taylor series about a for $y = \ln x$ and the range of x where this converges.

(b) Hence find $\ln 12$ to 5 decimal places, given that $\ln 10 = 2.302585$.

6. Given the following differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

where $y(0) = 1$ and $\frac{dy}{dx}(0) = 0$:

- (a) *Without trying to solve the equation*, find the values of the second, third and fourth derivatives of y at $x = 0$.
- (b) Show that $y = e^{-x}(1+x)$ agrees with the equation.
- (c) Write the first four non-zero terms of the MacLaurin series for y .
- (d) If the series is written as follows (using only non-zero terms):

$$y = 1 + \sum_{r=1}^{\infty} u_r$$

Find u_r in terms of r and x (you need not prove your answer for $r > 3$).

(e) Investigate the convergence of the series derived in part (d), for all real x .

7. This question concerns the following differential equation:

$$\frac{dy}{dt} = e^y$$

where $y(0) = 1$.

(a) Without solving for y , find the first five non-zero terms of the MacLaurin series for y as a function of time t .

(b) Given that the solution is $y = -\ln(e^{-1} - t)$, use this to confirm your answer to (a).

(c) If the series is written as follows:

$$y = 1 + \sum_{r=1}^{\infty} u_r$$

Find u_r in terms of r and t (you need not prove your answer for $r > 4$).

(d) Test the series derived in part (c) for convergence, given $t \geq 0$.

8. Find the first three non-zero terms of the MacLaurin series for y given:

$$\frac{d^2y}{dx^2} = xy$$

where $y(0) = 1$ and $\frac{dy}{dx}(0) = 0$.

9. Using a method derived from Taylor series, show that the proportional error in the *product* of two quantities, is approximately equal to the *sum* of the proportional errors in the two quantities themselves.

10. The value of g - the acceleration due to gravity - is calculated from the period T of a pendulum of length l using the formula:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

If T is measured 1% too large and l 3% too large;

(i) Using a method derived from Taylor series, find the approximate percentage error in the calculated value of g , and

(ii) compare with the actual percentage error.

11. The efficiency E of a petrol engine is given by:

$$E = 100 \left[1 - \left(\frac{1}{R} \right)^{0.25} \right]$$

where R is the expansion ratio. If R is increased by 1%, use a method derived from Taylor series to show that E is increased by about

$$\frac{1}{4} \left[1 - \frac{E}{100} \right]$$

Solutions

1. (a) $u_r = 1/(3^{r-1})$; convergent (b) $u_r = 1/(2r - 1)$; divergent
- (c) $u_r = (r - 1)/(r + 1)$; divergent (d) $u_r = x^r/r!$; convergent for all x
- (e) $u_r = r!/(x^{r-1})$; divergent for all x
- (f) $u_r = x^r/r^x$; convergent for $-1 < x < 1$ divergent for $x \geq 1$ and $x \leq -1$
- (g) $u_r = r/2^r$; convergent (h) $u_r = (r + 1)/r^2$; divergent
- (i) $u_r = (-1)^{r-1}x^r/r$; convergent for $-1 < x \leq +1$, divergent for $x > +1$ and $x \leq -1$
2. $1 + 4x + 6x^2 + 4x^3 + x^4$
3. (a) $1 - x^2 + x^4/3$ (b) $x^2 - x^3/2 + x^4/3$ (c) $1 - x^2 + x^4/2$
4. Parts (a) and (b) are the same, subject to choice of constant of integration
5. (a) $\ln x = \ln a + \sum_{r=1}^{\infty} (-1)^{r-1}(x - a)^r/(ra^r)$; convergent for $0 < x \leq 2a$ (b) $\ln 12 = 2.48491$
6. (a) $y'' = -1$, $y''' = 2$, $y^{iv} = -3$ (c) $y = 1 - x^2/2! + 2x^3/3! - 3x^4/4! + \dots$
- (d) $u_r = (-1)^r r x^{r+1}/(r + 1)!$ (e) convergent for all x
7. (a) $y = 1 + et + (et)^2/2 + (et)^3/3 + (et)^4/4 + \dots$
- (c) $u_r = (et)^r/r$ (d) convergent for $0 \leq t < 1/e$; divergent for $t \geq 1/e$
8. $y = 1 + x^3/6 + x^6/180 + \dots$