

## 2M1 Maths Tutorial: Series

1. Find a general expression  $u_r$  for the  $r^{\text{th}}$  term in the following infinite series (starting with  $r = 1$ ), and check for convergence.

- (a)  $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$
- (b)  $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$
- (c)  $0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3} + \dots$
- (d)  $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
- (e)  $1 + \frac{2}{x} + \frac{6}{x^2} + \frac{24}{x^3} + \dots$
- (f)  $x + \frac{x^2}{2^x} + \frac{x^3}{3^x} + \frac{x^4}{4^x} + \dots$
- (g)  $\frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{5}{32} + \frac{3}{32} + \frac{7}{128} + \dots$
- (h)  $2 + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \dots$
- (i)  $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

2. Find the MacLaurin series for  $(1+x)^4$ , and show that this is the same as its binomial expansion.

3. Find the first three non-zero terms of the MacLaurin series for the following:

- (a)  $\cos^2 x$
- (b)  $x \ln(x+1)$
- (c)  $e^{-x^2}$

4. Given  $f(x) = \ln x$ :

- (a) Integrate  $f(x)$  then expand the result as a  $3^{\text{rd}}$  order Taylor polynomial about  $x = 1$ .
- (b) Expand  $f(x)$  about  $x = 1$ , integrate the result, showing this to be consistent (to third order) with the answer to part (a).

5. (a) Find the Taylor series about  $a$  for  $y = \ln x$  and the range of  $x$  where this converges.

(b) Hence find  $\ln 12$  to 5 decimal places, given that  $\ln 10 = 2.302585$ .

6. Given the following differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

where  $y(0) = 1$  and  $\frac{dy}{dx}(0) = 0$ :

- (a) *Without trying to solve the equation*, find the values of the second, third and fourth derivatives of  $y$  at  $x = 0$ .
- (b) Show that  $y = e^{-x}(1+x)$  agrees with the equation.
- (c) Write the first four non-zero terms of the MacLaurin series for  $y$ .
- (d) If the series is written as follows (using only non-zero terms):

$$y = 1 + \sum_{r=1}^{\infty} u_r$$

Find  $u_r$  in terms of  $r$  and  $x$  (you need not prove your answer for  $r > 3$ ).

(e) Investigate the convergence of the series derived in part (d), for all real  $x$ .

7. This question concerns the following differential equation:

$$\frac{dy}{dt} = e^y$$

where  $y(0) = 1$ .

(a) Without solving for  $y$ , find the first five non-zero terms of the MacLaurin series for  $y$  as a function of time  $t$ .

(b) Given that the solution is  $y = -\ln(e^{-1} - t)$ , use this to confirm your answer to (a).

(c) If the series is written as follows:

$$y = 1 + \sum_{r=1}^{\infty} u_r$$

Find  $u_r$  in terms of  $r$  and  $t$  (you need not prove your answer for  $r > 4$ ).

(d) Test the series derived in part (c) for convergence, given  $t \geq 0$ .

8. Find the first three non-zero terms of the MacLaurin series for  $y$  given:

$$\frac{d^2y}{dx^2} = xy$$

where  $y(0) = 1$  and  $\frac{dy}{dx}(0) = 0$ .

9. Using a method derived from Taylor series, show that the proportional error in the *product* of two quantities, is approximately equal to the *sum* of the proportional errors in the two quantities themselves.

10. The value of  $g$  - the acceleration due to gravity - is calculated from the period  $T$  of a pendulum of length  $l$  using the formula:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

If  $T$  is measured 1% too large and  $l$  3% too large;

(i) Using a method derived from Taylor series, find the approximate percentage error in the calculated value of  $g$ , and

(ii) compare with the actual percentage error.

11. The efficiency  $E$  of a petrol engine is given by:

$$E = 100 \left[ 1 - \left( \frac{1}{R} \right)^{0.25} \right]$$

where  $R$  is the expansion ratio. If  $R$  is increased by 1%, use a method derived from Taylor series to show that  $E$  is increased by about

$$\frac{1}{4} \left[ 1 - \frac{E}{100} \right]$$

### Solutions

1. (a)  $u_r = 1/(3^{r-1})$ ; convergent (b)  $u_r = 1/(2r - 1)$ ; divergent
- (c)  $u_r = (r - 1)/(r + 1)$ ; divergent (d)  $u_r = x^r/r!$ ; convergent for all  $x$
- (e)  $u_r = r!/(x^{r-1})$ ; divergent for all  $x$
- (f)  $u_r = x^r/r^x$ ; convergent for  $-1 < x < 1$  divergent for  $x \geq 1$  and  $x \leq -1$
- (g)  $u_r = r/2^r$ ; convergent (h)  $u_r = (r + 1)/r^2$ ; divergent
- (i)  $u_r = (-1)^{r-1}x^r/r$ ; convergent for  $-1 < x \leq +1$ , divergent for  $x > +1$  and  $x \leq -1$
2.  $1 + 4x + 6x^2 + 4x^3 + x^4$
3. (a)  $1 - x^2 + x^4/3$  (b)  $x^2 - x^3/2 + x^4/3$  (c)  $1 - x^2 + x^4/2$
4. Parts (a) and (b) are the same, subject to choice of constant of integration
5. (a)  $\ln x = \ln a + \sum_{r=1}^{\infty} (-1)^{r-1}(x - a)^r/(ra^r)$ ; convergent for  $0 < x \leq 2a$  (b)  $\ln 12 = 2.48491$
6. (a)  $y'' = -1$ ,  $y''' = 2$ ,  $y^{iv} = -3$  (c)  $y = 1 - x^2/2! + 2x^3/3! - 3x^4/4! + \dots$
- (d)  $u_r = (-1)^r r x^{r+1}/(r + 1)!$  (e) convergent for all  $x$
7. (a)  $y = 1 + et + (et)^2/2 + (et)^3/3 + (et)^4/4 + \dots$
- (c)  $u_r = (et)^r/r$  (d) convergent for  $0 \leq t < 1/e$ ; divergent for  $t \geq 1/e$
8.  $y = 1 + x^3/6 + x^6/180 + \dots$