Revision Notes On Series C.T.J. Dodson, Department of Mathematics

Convergence Of Series

A **sequence** of numbers is an indexed list of the form $u_1, u_2, u_3, \cdots, u_n, \cdots$. Sequences typically arise as the solution to some recurrence relation. It is of interest to know if the u_n tend to a definite limit ℓ as n tends to infinity. If so, then we write $\lim_{n\to\infty} u_n = \ell$ or, briefly, just $u_n \to \ell$. It matters little if the sequence is numbered from $k = 0$ or $k = 1$; what is usually important is the behaviour for large values of k.

If we add up the first n terms in a sequence we get a **series** such as

$$\text{Sum of first n terms} = \sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n$$

and then we need to know if this sum is finite when we let the number of terms tend to infinity; if the sum is finite, then we say the series is **convergent** otherwise it is **divergent**. Three important series should be remembered and are often used in testing others (the first one is surprisingly divergent):

$$u_n = \frac{1}{n}\ (\textbf{Harmonic Series}): \quad \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \quad \textbf{Divergent to } +\infty$$

$$u_n = \frac{1}{n^2}\ (\textbf{Quadratic Series}): \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots \quad \textbf{Convergent to } \frac{\pi^2}{6}$$

$$u_n = \frac{1}{n!}\ (\textbf{Exponential Series}): \quad \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots \quad \textbf{Convergent to } e$$

Given an infinite series $u_1 + u_2 + \cdots + u_n + \cdots$, here is an outline procedure to test its convergence:

1 If $\lim_{n\to\infty} u_n \neq 0$ then series divergent. If $\lim_{n\to\infty} u_n = 0$, go to **2**.

2 If $\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \ell$, then series is: **convergent if** $\ell < 1$; **divergent if** $\ell > 1$; if $\ell = 1$ then go to **3**.

3 If $\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \ell$, then the result is indeterminate, except for the special case of **alternating series:**

If for all n, u_n and u_{n+1} are alternating in sign and $\lim_{n\to\infty} |u_n| = 0$, then series convergent. Example:

$$u_n = \frac{(-1)^n}{n}\ (\textbf{Alternating Series}): -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^n}{n} + \cdots \quad \textbf{Convergent to } -\log_{\mathbf{e}} \mathbf{2}$$

In general, it is quite difficult to find a simple expression for the limit of a convergent series.

Power Series

Generalizing polynomials, a large class of functions can be expressed as a convergent series of the form

$$f(x) = \sum_{n=0}^{n=\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

for some constant coefficients a_n and some range of values $-R < x < R$. Important example:

$$\textbf{Exponential Function}: \ e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \quad \textbf{Convergent for all real } x.$$

There is a special case for functions that have derivatives of all orders. Then, using $f^{(n)}(a)$ to represent the n^{th} derivative of f evaluated at a we have

$$\textbf{Taylor Series}: \quad f(x+a) = f(a) + f'(a)\,x + \frac{f''(a)\,x^2}{2!} + \cdots + \frac{f^{(n)}(a)\,x^n}{n!} + \cdots$$

By the procedure given above, the Taylor series is convergent for all $|x| < R$ where $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. The values $x = \pm R$ have to be tested individually for convergence; R is the **Radius of Convergence**.

In the case that $a = 0$, then a Taylor Series is called a **MacLaurin Series**.

Note that some functions cannot be expressed as a power series. For example, $f(x) = |x|$ is defined for all real x but does not have a power series expansion for this range of x. It is clear why it cannot have a MacLaurin series: it is not differentiable at $x = 0$ because there is a corner in the graph there.