2M1 Tutorial: Partial Differential Equations

1. Calculate u_x, u_y, u_{xx}, u_{xy} and u_{yy} for the following:
(a) $u = x^2 - y^2$
(b) $u = e^x \cos y$
(c) $u = \ln(x^2 + y^2)$
Hence show that all three functions are possible solutions of the PDE: $u_{xx} + u_{yy} = 0$.

2. Find the general solution of: $\frac{\partial^2 y}{\partial x \partial t} = 2$.

3. Which of the following could be solved by separation of variables?
(a) $u_{xy} + 2u = 0$ (b) $x^2 u_{xx} + y u_{yy} = 0$ (c) $2u_{xx} + 3u_{xy} - u_y = 0$
(d) $u_{xx} + 2uxy + u_{yy} = 0$ (e) $u_{xx} + u_{yy} + u_x + u_y = 0$.

4. By using a trial solution $u(x,t) = X(x)T(t)$ and separation of variables, find solutions of the PDEs:
(a) $2u_x - u_t = 0$ where $u(0,t) = 2e^{-4t}$
(b) $u_x + u = u_t$ where $u(x,0) = 4e^{-3x}$
(c) $u_{xx} + u_{tt} = 0$ where: $u(0,t) = \sin t$ and $u(\infty, t) = 0$.

5. Find general solutions to the following, using characteristic lines, and then show that the solutions to question 4 parts (a) and (b) are compatible with your results:
(a) $2u_x - u_t = 0$ Hint: change variables to $\epsilon = x,\ \phi = -\frac{1}{2}x - t$ (b) $u_x + u = u_t$ Hint: change variables to $\epsilon = x,\ \phi = -x - t$.

6. Classify the following as hyperbolic, elliptic or parabolic:
(a) $u_{xx} = u_{tt}/c^2$ (wave equation)
(b) $u_t = c^2 u_{xx}$ (heat equation)
(c) $u_{xx} + u_{yy} = 0$ (Laplace′s equation)
(d) $9u_{tt} - 6u_t + u_{xx} = 0$.

7. (a) Show that the general solution of the wave equation: $u_{xx} = u_{tt}/c^2$ is:

$$u(x,t) = f(x + ct) + g(x - ct),$$

where f and g are arbitrary functions. Hint: for a hyperbolic equation $au_{xx} + 2hu_{xt} + bu_{tt} = 0$, solutions λ_1, λ_2 of associated quadratic $a + 2h\lambda + b\lambda^2 = 0$, then general solution of form $u(x,t) = f(x + \lambda_1 t) + g(x + \lambda_2 t)$.
(b) (i) Show that a possible solution of the wave equation for the displacement of a stretched vibrating string which also fits the boundary conditions of being fixed at $x = 0$ and $x = l$ is:

$$u(x,t) = \sin\left(\frac{\pi x}{l}\right)\left[\cos\left(\frac{c\pi t}{l}\right) + \sin\left(\frac{c\pi t}{l}\right)\right]$$

(ii) Show that the solution in (i) is compatible with the general solution as given in part (a).

8. Find general solutions to the following:
(a) $u_{xx} + 2u_{xy} + u_{yy} = 0$ Hint: for a parabolic equation $au_{xx} + 2hu_{xy} + bu_{yy} = 0$, only one solution of associated quadratic $a + 2h\lambda + b\lambda^2 = 0$, so general solution of form $u(x,y) = f(x + \lambda y) + (rx + sy)g(x + \lambda y)$.
(b) $2u_{xx} - 3u_{xy} + u_{yy} = 0$.

Answers

1. (i) $u_x = 2x$, $u_y = -2y$, $u_{xx} = 2$, $u_{xy} = 0$, $u_{yy} = -2$
(ii) $u_x = e^x \cos y$, $u_y = -e^x \sin y$, $u_{xx} = e^x \cos y$, $u_{xy} = -e^x \sin y$, $u_{yy} = -e^x \cos y$
(iii) $u_x = 2x/(x^2 + y^2)$, $u_y = 2y/(x^2 + y^2)$, $u_{xx} = 2(y^2 - x^2)/(x^2 + y^2)^2$,
$u_{xy} = -4xy/(x^2 + y^2)^2$, $u_{yy} = 2(x^2 - y^2)/(x^2 + y^2)^2$.
2. $y(x,t) = 2xt + f(t) + g(x)$ where f and g are arbitrary functions.
3. (a) yes (b) yes (c) yes (d) no (e) yes.
4. (a) $u(x,t) = 2e^{-2x-4t}$ (b) $u(x,t) = 4e^{-(3x+2t)}$ (c) $u(x,t) = e^{-x} \sin t$.
5. (a) $u(x,t) = f(-x/2 - t)$ (b) $u(x,y) = e^{-x + f(-x-t)}$ where f is an arbitrary function.
6. (i) hyperbolic (ii) parabolic (iii) elliptic (iv) parabolic.
7. (a) Solve as for a hyperbolic equation (b) (ii) use trig. identities from the formula book.
8. (a) $u(x,y) = f(x - y) + (rx + sy)g(x - y)$ (b) $u(x,y) = f(x + 2y) + g(x + y)$
(where f and g are arbitrary functions, r and s constants)