Revision Notes On PDEs C.T.J. Dodson, Department of Mathematics

Partial Differential Equations

Ordinary differential equations involve finding a function (or a set of functions) of one independent variable but partial differential equations are for functions of two or more variables. Examples of such models are the heat equation for the evolution of the temperature distribution in a body, the wave equation for the motion of a wavefront, the flow equation for the flow of fluids and Laplace's equation for an electrostatic potential or elastic strain field. In such cases we need to have not only the initial conditions, but also **boundary conditions** for the region in which the model applies; thus we have to solve **boundary value problems**. As with ODEs, we call a PDE *homogeneous* if a linear combination of derivatives is equal to zero—and then a linear combination of solutions is another solution. Here are typical examples of the commonest types, for the simplest case—just two independent variables (x, t or x, y—it is easy to see how they would generalize to more variables x, y, z, t)

$$\begin{aligned}
\text{Flow Equation}\quad c\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t} &= 0, \quad \text{given initial or boundary values for } u \qquad (1)\\
\text{Heat Equation}\quad c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t} &= 0, \quad \text{given initial or boundary values for } u \qquad (2)\\
\text{Wave Equation}\quad c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2} &= 0, \quad \text{given initial or boundary values for } u \qquad (3)\\
\text{Laplace's Equation}\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} &= 0, \quad \text{given boundary values for } v. \qquad (4)
\end{aligned}$$

Solutions

Solutions to practical models using these equations are usually very difficult to obtain analytically and computers are used to obtain numerical approximate solutions by standard iterative procedures. In some simple cases, solutions can be found in terms of sums and products of elementary functions. Try to match the following examples of functions $u(x,t)$ or $u(x,y)$ to the above equations and suitable conditions:

$$3e^{-at}\sin\sqrt{a/b}x,\ \ e^{-ax}\cos ay,\ \ \log(x^2+y^2),\ \ e^{-x}\cos(t-x),\ \ \sin(ax)\sinh(ay),\ \ e^{-a^2t}\sin(ax),\ \ 2\cos(ax)\cos(act)$$

Method of characteristics

For a problem of type (1), note that u is constant along a curve in (x,t)-space given by $\frac{dx}{dt}=c$ using the method of characteristics and it follows that $u(x,t)$ is a function of $(x-ct)$ only. Here the characteristics are the lines $t=\frac{1}{c}x+constant$, on which u is constant. Consider next the case that the flow equation is not homogeneous, for example

$$c\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=kt, \quad \text{given initial or boundary values for } u. \qquad (5)$$

Then we find that there is a particular solution $u_{PI}(x,t)=\frac{1}{2}kt^2$, and so a general solution of (5) is of the form $u(x,t)=\frac{1}{2}kt^2+f(x-ct)$, for some function f; then $f(x-ct)$ represents a wave travelling along the x-axis at constant speed c with no change in its shape. The shape of the waveform may be determined from the given initial or boundary conditions.

D'Alembert's solution

The wave equation (3) has the D'Alembert solution $\phi(x-ct)+\psi(x+ct)$, for suitable choices of the functions ϕ and ψ to suit the given conditions. Again we have waves travelling at constant speed c, in both directions along the x-axis.

Separation of variables

A general method for attempting to solve PDEs is to suppose that the solution function u is a product of functions, each one depending on one only of the independent variables. This converts the PDE into two (or more) ODEs which may be soluble. For example, consider the heat equation (2),

$$\text{Substituting}\quad u(x,t)=F(x)G(t)\quad \text{in}\quad c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=0\quad \text{gives}\quad \frac{1}{c^2G}\frac{dG}{dt}=\frac{1}{F}\frac{d^2F}{dx^2}$$

In the ODE, each side depends on a different single variable only, hence both sides must be equal to the same constant. That yields two ODEs to solve subject to the given conditions for the problem. Can you write down the corresponding ODEs for solution of the wave equation by separation of variables?