2M1 Tutorial: Ordinary Differential Equations

1. Without solving (unless you want extra practice!), show that:
(a) $y = e^x(1 + x)$ is a solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$
(b) $y = Ae^x + Be^{-x}$ is a solution of $\frac{d^2y}{dx^2} - y = 0$
(c) $y = Ae^x + Be^{2x} + x^2e^x$ is a solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^x(1 - x)$
2. Given the equations: (i) $\ddot{y} + \omega^2 y = 0$ (ii) $\ddot{y} - \omega^2 y = 0$:
(a) Find the general solution of each.
(b) Show that the solution to (ii) can be expressed in a similar way to (i), but using hyperbolic functions
3. Solve the following:
a) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ (general solution)
b) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$ (general solution)
c) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ (general solution)
d) $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 36y = 0$ (general solution)
e) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 9y = 0$ (general solution)
f) $2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$ where $y(0) = \sqrt{2}$ and $\frac{dy}{dx}(0) = 0$
g) $\frac{d^2y}{dx^2} + 16y = 0$ (general solution)
h) $\frac{d^2y}{dx^2} + 7y = 0$ (general solution)
i) $\frac{d^2y}{dx^2} - 9y = 0$ (general solution)
j) $\frac{d^2y}{dx^2} - 4y = 0$ where $y(0) = 0$ and $\frac{dy}{dx}(0) = 2$
k) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$ (general solution)
l) $\frac{d^2y}{dx^2} + 4(y - \frac{dy}{dx}) = 0$ (general solution)
m) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$ where $y(0) = 4$ and $\frac{dy}{dx}(0) = 0$

4. Solve the equation for constant acceleration; $\ddot{y} = a$ using the method for a general second order linear equation with constant coefficients.

5. Show that:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

can be expressed as

$$(D - \lambda_1)(D - \lambda_2)y = 0$$

where λ_1 and λ_2 are the solutions to the auxiliary equation, and D represents the differential operator, '$\frac{d}{dx}$'.

6. Given an inhomogeneous equation of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

show that $y = u_c + u_p$ where u_c and u_p are the complementary function and particular integral respectively.
7. Prove that $A\cos nx + B\sin nx$ can be written in the form $\sqrt{A^2 + B^2}\sin(nx + \alpha)$ where $\tan\alpha = A/B$. What is the engineering significance of this alternative form? Find the solution of the equation $\frac{d^2y}{dx^2} + 25y = 0$ in this alternative form, given $A = B = 1$.
8. What would you normally take as the assumed particular integral u_p in each of the following cases: (a) $f(x) = 2x - 3$ (b) $f(x) = e^{5x}$ (c) $f(x) = \sin 4x$ (d) $f(x) = 3 - 5x^2$
(e) $f(x) = 27$ (f) $f(x) = 5\cosh 4x$ (g) $f(x) = x + 2e^x$
9. Find general solutions for the following:
(a) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 24$
(b) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2\sin 4x$
(c) $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 4e^{5x}$
(d) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{-3x}$
(e) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 8e^{4x}$

10. Solve the following:

(a) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2e^{-2x}$; where $y(0) = 1$ and $\frac{dy}{dx}(0) = -2$
(b) $\ddot{x} + 4\dot{x} + 3x = e^{-3t}$; where $x(0) = 1/2$ and $\dot{x}(0) = -2$
(c) $\ddot{x} - 3\dot{x} + 2x = \sin t$; where $x(0) = 0$ and $\dot{x}(0) = 1$
(d) $\frac{d^2y}{dx^2} = 3\sin x - 4y$; where $y(0) = 0$ and $\frac{dy}{dx}(\pi/2) = 1$
(e) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$; where $y(0) = 0$ and $\frac{dy}{dx}(0) = 1$
(f) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}$; where $y(0) = 0$ and $\frac{dy}{dx}(0) = 1$
11. The equation of motion of a pendulum subjected to forced oscillations is:

$$Mk^2\ddot{\theta} = Mah\omega^2 \cos\theta \sin\omega t - Mgh\sin\theta$$

where θ is the angle of inclination of the pendulum to the downward vertical at time t, and the parameters M, k, a, h, ω, g are constants.
Show that for small values of θ this equation reduces to a linear, second order differential equation, and thereby show that if the pendulum starts from rest in the downward vertical position, θ is given by:

$$\theta(t) = \frac{ah\omega^2}{gh - \omega^2k^2}\left[\sin\omega t - \frac{\omega k}{\sqrt{gh}}\sin\left(\frac{\sqrt{gh}}{k}t\right)\right]$$

12. For a horizontal cantilever of length l, with load w per unit length, the equation of bending is $EIy'' = (1/2)w(l-x)^2$, where E, I, w and l are constants. If $y = 0$ and $y' = 0$ at $x = 0$, find y in terms of x. Hence find the value of y when $x = l$.
13. Solve the following:
(a) $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$ (general solution)
(b) $\frac{d^4y}{dx^4} - y = 0$; where y and its first two derivatives are all zero at $x = 0$, and $\frac{d^3y}{dx^3}(0) = 4$
(c) $\frac{d^3y}{dt^3} + \frac{dy}{dt} = 12\sin 2t$ (general solution)

Solutions

2. (a) (i) $y = A\cos\omega t + B\sin\omega t$ (ii) $y = Ae^{\omega t} + Be^{-\omega t}$
(b) $y = A\cosh\omega t + B\sinh\omega t$ (with different A and B)
3. (a) $Ae^{-2x} + Be^{-3x}$ (b) $Ae^{3x} + Be^{4x}$ (c) $(Ax+B)e^{-3x}$
(d) $(Ax+B)e^{6x}$ (e) $e^{-2x}(A\cos\sqrt{5}x + B\sin\sqrt{5}x)$
(f) $e^{-x}(\sqrt{2}\cos\frac{x}{\sqrt{2}} + 2\sin\frac{x}{\sqrt{2}})$ (g) $A\cos 4x + B\sin 4x$ (h) $A\cos\sqrt{7}x + B\sin\sqrt{7}x$
(i) $Ae^{3x} + Be^{-3x}$ or $A\cosh 3x + B\sinh 3x$ (j) $\sinh 2x$ or $1/2(e^{2x} - e^{-2x})$ (k) $Ae^{-4x} + Be^{x}$
(l) $(Ax+B)e^{2x}$ (m) $e^{x}(4\cos 2x - 2\sin 2x)$
7. $y = \sqrt{2}\sin(5x + \pi/4)$ or $y = \sqrt{2}\sin[5(x + \pi/4)]$
8. (a) $mx + n$ (b) me^{5x} (c) $m\cos 4x + n\sin 4x$ (d) $lx^2 + mx + n$
(e) m (f) $m\cosh 4x + n\sinh 4x$ (g) $lx + m + ne^{x}$
9. (a) $Ae^{2x} + Be^{3x} + 4$ (b) $Ae^{2x} + Be^{3x} + (1/25)(2\cos 4x - \sin 4x)$
(c) $e^{-7x}(Ax+B) + e^{5x}/36$ (d) $Ae^{-3x} + Be^{2x} - (x/5)e^{-3x}$
(e) $Ae^{2x} + e^{4x}(B + 4x)$
10. (a) $e^{-2x}(2 - \cos x)$ (b) $(1/2)e^{-3t}(1-t)$
(c) $-(3/2)e^{t} + (6/5)e^{2t} + (1/10)(3\cos t + \sin t)$
(d) $-(1/2)\sin 2x + \sin x$ (e) $(-x-6)e^{-x} + x^2 - 4x + 6$
(f) $(x + x^2/2)e^{-x}$
12. $y(l) = wl^4/(8EI)$
13. (a) $Ae^{-x} + Be^{3x} + Ce^{-2x}$ (b) $e^{x} - e^{-x} - 2\sin x$ (c) $A + B\cos t + C\sin t + 2\cos 2t$