Revision Notes On ODEs
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Differential Equations

Algebraic equations we encounter in engineering problems usually have numbers as solutions. On the other hand, many real processes are observed to depend on the rates of change of quantities as well as their values. So, mathematical models of real processes often give rise to **differential equations** and their solutions are **functions**. Most differential equations have no solution in terms of elementary functions like polynomials, trig and exponential functions. Indeed, many special functions are defined to be the solutions of particular differential equations that first arose in practical problems.

Ordinary Differential Equations (ODEs)

An ODE is **linear** if, given any two solutions, then **the sum of the solutions is also a solution**; in this case, we have no nonlinear terms like $y\,y'$, or $(y')^2$ in the ODE. Surprisingly, many dynamical problems of practical interest can be modelled by second order differential equations, of the type where the left hand side is a linear combination of derivatives of y (including the 0^{th}) with constant coefficients:

$$a\,y'' + b\,y' + c\,y = f(x) \quad \text{with initial conditions} : y(x_0) = y_0,\ y'(x_0) = v_0 \text{ given constants} \quad (1)$$

Here $f(x)$ represents a 'forcing term' and we seek a solution function $y(x)$, normally, for the largest available range of x including the initial condition at $x = x_0$ (often, $x_0 = 0$. This independent variable x can represent time or some other quantity. We call (1) an **initial value problem**. A related problem is when we are given two values of y (at x_1 and x_2, say) that must be fitted (instead of $y(0)$ and $y'(0)$):

$$a\,y'' + b\,y' + c\,y = f(x) \quad \text{with boundary conditions} : y(x_1) = y_1,\ y(x_2) = y_2 \text{ given constants} \quad (2)$$

This is called a **boundary value problem**. Both types of problem are solved similarly, in three steps:

Solution of constant coefficient ODEs (1) and (2):

(i) Complementary Function y_{CF}

First consider the so-called **homogeneous**, or **complementary equation:** $a\,y''+b\,y'+c\,y = 0$. Its **auxiliary equation** is the quadratic $a\,m^2 + b\,m + c = 0$, which in general has two solutions m_1, m_2. Three cases arise:
If m_1, m_2 **real and distinct:** $y_{CF} = A\,e^{m_1 x} + B\,e^{m_2 x}$ **solves** $a\,y'' + b\,y' + c\,y = 0$.
If m_1, m_2 **real and the same:** $y_{CF} = A\,e^{m_1 x} + B\,x\,e^{m_1 x}$ **solves** $a\,y'' + b\,y' + c\,y = 0$.
If $m = \alpha \pm i\beta$ **complex conjugate:** $y_{CF} = e^{\alpha x}(A\,\cos\beta x + B\,\sin\beta x)$ **solves** $a\,y'' + b\,y' + c\,y = 0$.
In each case we call the function y_{CF} the **complementary function** and since it has **two arbitrary constants**, A, B, it is the **general solution** of the second order homogeneous equation $a\,y'' + b\,y' + c\,y = 0$.

(ii) Particular Integral y_{PI}

To complete the solution of (1) or (2), we need to deal with the fact that the right hand side is not in fact zero. To do this we must find any function, y_{PI} say, which actually gives the correct right hand side. Then $y(x) = y_{CF}(x) + y_{PI}(x)$ will be a general solution of $a\,y'' + b\,y' + c\,y = f(x)$ because it has two arbitrary constants. The standard way is to **guess the form of** y_{PI}**, to be like** $f(x)$ (but introduce some constants sprinkled about it), **substitute**, then **adjust the constants to fit:**
If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, **guess:** $y_{PI} = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$.
If $f(x) = A\,e^{kx}$, **guess:** $y_{PI} = C\,e^{kx}$.
If a y_{PI} contains a term that is obtainable from the y_{CF}, then multiply it by x (or by x^2 if necessary).

(iii) Complete Solution $y = y_{CF} + y_{PI}$

To obtain the complete solution, take $y = y_{CF} + y_{PI}$ and substitute the initial or boundary conditions to solve for the arbitrary constants. Note: this **substitution must be done in the complete solution**.

Series Solution

We can develop a power series solution to a general (not necessarily linear nor constant coefficient) ODE, assuming that the solution has a Taylor series that is convergent near the initial conditions. The method is to substitute a general power series and try to solve for the coefficients. If we have initial data at $x = 0$, then we can quite easily obtain an approximation to a solution near this initial data.
Example: Series solution to $y'' + x\,y' + y = 0, \quad y(0) = 0,\ y'(0) = 1$
Rearranging, we can write $y'' = -x\,y' - y$ and obtain higher derivatives by differentiation. Then, we develop Maclaurin's series by substituting the values of these derivatives at $x = 0$:

$$y(x) = y(0) + y'(0)x + y''(0)/2! + \cdots \quad = x - \frac{2}{3!}x^3 + \frac{8}{5!}x^5 + \cdots$$