2M1 Tutorial: Laplace Transforms

1. Derive the transforms of the following:
(a) $\cos \omega t$ (suggestion: integration by parts)
(b) $1/\sqrt{t}$ (suggestion: substitution $u = \sqrt{st}$)
(c) $t \cos \omega t$ (suggestion: integration by parts)
You may need the following standard result for part (b):

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

2. Find the inverse transforms of the following:

$$(a)\frac{1}{(s+1)} \qquad (b)\frac{s}{(s^2+9)^2} \qquad (c)\frac{4}{(s+2)^4}$$

$$(d)\frac{2s+3}{(s+4)(s+5)} \qquad (e)\frac{1}{(s+1)(3s+1)} \qquad (f)\frac{s+2}{s^2+4s+13}$$

3. Solve the following differential equations, given $y(0) = 0$, and $\frac{dy}{dt}(0) = 1$ (where appropriate):
(a) $\frac{dy}{dt} + y = 1$ (b) $\frac{dy}{dt} + y = t$ (c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = 0$ (d) $\frac{d^2y}{dt^2} + 4\left(y - \frac{dy}{dt}\right) = 0$
(e) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = 0$ (f) $\frac{d^2y}{dt^2} + y = \cos t$ (g) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}$

4. The following are transforms $\tilde{y}(s)$ for systems subject to a sudden change in conditions. Find their inverses $y(t)$, and hence describe the behaviour of the function $y(t)$ before and after the change:

$$(a)\frac{2}{s} + \frac{e^{-3s}}{(s+4)} \qquad (b)\frac{s}{s^2-4} - \frac{e^{-s}(s^2-9)}{(s^2+9)^2} \qquad (c)\frac{12e^{-2s}}{s^4}$$

5. (a) Solve the following, given $y(0) = 0$, and $\frac{dy}{dt}(0) = 1$
$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2 + \delta(t-1)$
(b) (i) Solve the following, given $y(0) = 0$, and $\frac{dy}{dt}(0) = 1$
$\frac{d^2y}{dt^2} + y = \delta(t-\pi)$
(ii) Describe in words the effect of the impulse in part (i).
(c) Repeat part (b) (i) without using Laplace transforms.
(d) Solve the following, given $y(0) = 0$, and $\frac{dy}{dt}(0) = 0$
$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t} + 3\delta(t-1)$
(e) Solve the following, given $y(0) = 1$, and $\frac{dy}{dt}(0) = 1$
$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3\delta(t-1) + \delta(t-2)$

6. (a) Solve the following system of simultaneous differential equations given $x(0) = 0$; $y(0) = 1$ as follows:

(i) Find transforms of both equations. (ii) Solve the simultaneous *algebraic* equations thus formed, to get separate expressions for the transforms of each function. (iii) Invert the results of part (ii) to find x and y as functions of t.

$$\frac{dx}{dt} = -3x + y$$

$$\frac{dy}{dt} = 4x - 6y$$

(b) Using the same procedure as in part (a), solve the following system of differential equations given $x(0) = 0$; $y(0) = 1/2$:

$$\frac{dx}{dt} + \frac{dy}{dt} + y = 0$$

$$2\frac{dx}{dt} + \frac{dy}{dt} - y = e^{-t}$$

Answers

1. (a) $s/(s^2+\omega^2)$ (b) $\sqrt{\pi/s}$ (c) $(s^2-\omega^2)/(s^2+\omega^2)^2$
2. (a) e^{-t} (b) $(t\sin 3t)/6$ (c) $(2/3)t^3e^{-2t}$ (d) $7e^{-5t}-5e^{-4t}$ (e) $(1/2)(e^{-t/3}-e^{-t})$ (f) $e^{-2t}\cos 3t$
3. (a) $1-e^{-t}$ (b) $e^{-t}+t-1$ (c) $(e^t-e^{-4t})/5$ (d) te^{2t}
(e) $(e^{2t}\sin 2t)/2$ (f) $\sin t(1+t/2)$ (g) $te^{-t}(1+t/2)$
4. (a) $y=2$ for $t<3$; $y=2+e^{-4(t-3)}$ for $t\geq 3$
(b) $y=\cosh 2t$ for $t<1$; $y=\cosh 2t+(t-1)\cos(3(t-1))$ for $t\geq 1$
(c) $y=0$ for $t<2$; $y=2(t-2)^3$ for $t\geq 2$
5. (a) $1-e^{-t}+H_1(t)[e^{-(t-1)}-e^{-2(t-1)}]$
(b) (i) $\sin t+H_\pi(t)\sin(t-\pi)$ (ii) Oscillation stops after 1/2 cycle
(c) Use auxiliary equation etc. as in earlier method, with additional velocity due to the impulse contributing to new starting conditions at $t=\pi$.
(d) $t^2e^{-t}/2+3H_1(t)[(t-1)e^{-(t-1)}]$
(e) $e^{2t}(1-t)+3H_1(t)[(t-1)e^{2(t-1)}]+H_2(t)[(t-2)e^{2(t-2)}]$
6. (a) $x=(e^{-2t}-e^{-7t})/5$; $y=(e^{-2t}-4e^{-7t})/5$
(b) $x=(2/3)(1-e^{-3t})$; $y=e^{-3t}-e^{-t}/2$